Common Fixed Point Theorems for S-Weakly Commuting, S-Compatible and RS-Weakly Commuting Mappings of Complete S-Fuzzy Metric Spaces

M. S. Rathore, Deepak Singh and Naval Singh

deepak.singh.2006@indiatimes.com

Abstract. In this paper we prove common fixed point theorem for S-Weakly commuting, S-Compatible and RS-weakly commuting maps in S-Fuzzy metric spaces.

Mathematics Subject Classification: 54H25, 47H10

Keywords: S-Fuzzy metric spaces, S-Weakly commuting maps, S-Compatible and RS-weakly commuting maps

1. INTRODUCTION

2. PRELIMINARIES

**Definition 2.1 [1].** The 3-tuple \((X, S, \ast)\) is said to be a S-fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) a continuous t-norm and \(S\) is a fuzzy set on \(X^3 \times (0, \infty)\), satisfying the following conditions

(i) \(S(x, y, z, t) > 0\),

(ii) \(S(x, y, z, t) = 1\) if and only if \(x = y = z\) (coincidence),

(iii) \(S(x, y, z, t) = S(y, z, x, t) = S(z, y, x, t) = \ldots\) (Symmetry),

(iv) \(S(x, y, z, r + s + t) \geq S(x, y, w, r) \ast S(x, w, z, s) \ast S(w, y, z, t)\) (tetrahedral inequality)

(v) \(S(x, y, z, \cdot) : (0, \infty) \rightarrow [0,1]\) is continuous for all \(x, y, z, w \in X\) and \(r, s, t > 0\).

**Definition 2.2 [1].** A sequence \(\{x_n\}\) in a S-fuzzy metric space \((X, S, \ast)\) is a Cauchy sequence if and only if for each \(\varepsilon > 0, t > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[S(x_n, x_m, x_p, t) > 1 - \varepsilon\]

for all \(n, m, p \geq n_0\).

**Definition 2.3 [1].** A sequence \(\{x_n\}\) in a fuzzy metric space \((X, M, \ast)\) converges to \(x\) if and only if for each \(\varepsilon > 0, t > 0\), there exists \(n_0 \in \mathbb{N}\) such that

\[M(x_n, x, t) > 1 - \varepsilon\]

for all \(n \geq n_0\).

**Definition 2.4 [1].** A S-fuzzy metric space in which every Cauchy sequence is a convergent sequence, is called a complete S-fuzzy metric space.

Geometrically, \(S(x, y, z, t)\) represents the fuzzy perimeter of the triangle whose vertices are the points \(x, y\) and \(z\) with respect to \(t > 0\).

**Definition 2.5 [7].** Let \(A\) and \(B\) be mappings from a metric space \((X, d)\) into itself. Then \(A\) and \(B\) are said to be weakly commuting mappings on \(X\) if

\[d(ABx, BAx) \leq d(Ax, Bx)\]

for all \(x \in X\).

**Definition 2.6 [3,4].** Let \(A\) and \(B\) be mappings from a metric space \((X, d)\) into itself. Then \(A\) and \(B\) are said to be compatible mappings on \(X\) if

\[\lim_{n \to \infty} d(ABx_n, BAx_n) = 0\]

where \(\{x_n\}\) is a sequence in \(X\) such that

\[\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = t\]

for some point \(t\) in \(X\).

**Definition 2.7 [2]** Two mappings \(f\) and \(g\) of a fuzzy metric space \((X, M, \ast)\) into itself are said to be weakly commuting if \(M(fgx, gfx, t) \leq M(fx, gx, t)\) for each \(x \in X\).

**Definition 2.8 [2].** Self mappings \(F\) and \(G\) of a fuzzy metric space \((X, M, \ast)\) are said to be compatible iff \(M(FGx_n, GFx_n, t) \rightarrow 1\) for all \(t > 0\), whenever \(\{x_n\}\) is a sequence in \(X\) such that \(Gx_n, Fx_n \rightarrow y\) for some \(y\) in \(X\).

Now we define S-weakly commuting maps and S-compatible maps in S-fuzzy metric space \((X, S, \ast)\)

**Definition 2.9.** Two self maps \(A\) and \(B\) of a S-fuzzy metric space \((X, S, \ast)\) are said to be S-weakly commuting if

\[S(ABx, BAx, y, t) \geq S(Ax, Bx, z, t)\]

where \(y = ABx\) or \(BAx\) and \(z = Ax\) or \(Bx\) for all \(x \in X\).
**Definition 2.10.** Two self mappings A and B of a S-fuzzy metric space 
\((X, S, *)\) are said to be S-compatible if

\[
\lim_{n \to \infty} S(ABx_n, BAx_n, z, t) = 1 \text{ where } z = ABx_n \text{ or } BAx_n, \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that } \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y, \text{ for some } y \in X.
\]

Clearly, commutativity implies S-weak commutativity and S-weak commutativity implies S-compatibility, but neither implication is reversible always. This can be seen in following examples.

**Example 2.1 -** Let \(X = [0, 1]\).
Define \(S(x, y, z, t) = \min\{M(x, y, t), M(y, z, t), M(z, x, t)\}\),

\[
M(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad d(x, y) = |x - y| \quad \forall \ x, y \in X.
\]

Also define self maps A and B of X by \(Ax = x^2\), \(Bx = x^2/2 \quad \forall \ x \in X\). Then we see that \(AB \neq BA\) and \(S(ABx_n, BAx_n, ABx_n, t) \geq S(Ax_n, Bx_n, Ax_n, t) \quad \forall \ x \in [0, 1]\).
This shows S-weak commutativity does not imply commutativity.

**Example 2.2 -** Let \(X = \mathbb{R}\)
Define \(S(x, y, z, t) = \min\{M(x, y, t), M(y, z, t), M(z, x, t)\}\) where

\[
M(x, y, t) = \frac{t}{t + d(x, y)} \quad \text{and} \quad d(x, y) = |x - y| \quad \forall \ x, y \in \mathbb{R}.
\]

Also define self maps A and B of X by \(Ax = x^2\), \(Bx = x^3/3 \quad \forall \ x \in \mathbb{R}\) and \(x_n = 1/n, n = 1, 2, 3 \ldots\)
Here \(\lim Ax_n = \lim Bx_n = 0 \in X\).
And \(S(ABx_n, BAx_n, ABx_n, t) \to 1 \text{ as } n \to \infty\). But \(S(ABx, BAx, ABx, t) \neq S(Ax, Bx, Ax, t) \) is not true for all \(x \in \mathbb{R}\) and \(AB \neq BA\).
Thus we see that A and B are S-compatible but neither commutative nor S-weakly commutative.
3. Common fixed point theorems for S-weakly commuting maps and S-compatible maps in complete S-fuzzy metric spaces.

We prove the following theorem for S-weakly commuting maps.

**Theorem 3.1.** Let A, B, P and T be self maps of a complete S-fuzzy metric space \((X, S, *)\) with t-norm * defined by \(a * b = \min \{a, b\}\), \(a, b \in [0, 1]\) satisfying the conditions

(i) \(A(X) \subseteq T(X), B(X) \subseteq P(X)\),

(ii) One of A, B, P or T is continuous,

(iii) (A, P) and (B, T) are S-weakly commuting pairs of maps,

(iv) for all \(x, y, z \in X, 0 < k < 1, t > 0\)

\[ S(Ax, By, z, kt) \geq \min \{S(Px, Ty, z, t), S(Ax, Ty, z, t), S(By, Px, z, t)\} \]

and

(v) \(S(x, y, z, t) \rightarrow 1\) as \(t \rightarrow \infty\)

Then A, B, P and T have a unique common fixed point in X.

**Proof:** Let \(x_0 \in X\) be arbitrary, construct a sequence \(\{y_n\}\) in X such that

\[ y_{2n+1} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n} = Px_{2n} = Bx_{2n-1}; \quad n = 0, 1, 2, \ldots \]

using (iv), we have

\[ S(y_1, y_2, y_m, kt) = S(Ax_0, Bx_1, y_m, kt) \]

\[ \geq \min \{S(Px_0, Tx_1, y_m, t), S(Ax_0, Tx_1, y_m, t), S(Bx_1, Px_0, y_m, t)\} \]

\[ = \min \{S(y_0, y_1, y_m, t), S(y_1, y_1, y_m, t), S(y_2, y_0, y_m, t)\} \]

\[ \geq \min \{S(y_0, y_1, y_m, t), S(y_1, y_2, y_m, t), S(y_0, y_2, y_m, t)\}. \]

This implies that

\[ S(y_1, y_2, y_m, kt) \geq S(y_0, y_1, y_m, t) \text{ or } S(y_0, y_2, y_m, t). \]

Further using (iv), we have

\[ S(y_2, y_3, y_m, kt) = S(Bx_1, Ax_2, y_m, kt) = S(Ax_2, Bx_1, y_m, kt) \]

\[ \geq \min \{S(Px_2, Tx_1, y_m, t), S(Ax_2, Tx_1, y_m, t), S(Bx_1, Px_2, y_m, t)\} \]

\[ = \min \{S(y_2, y_1, y_m, t), S(y_3, y_1, y_m, t), S(y_2, y_2, y_m, t)\} \]

\[ \geq \min \{S(y_1, y_2, y_m, t), S(y_1, y_3, y_m, t), S(y_2, y_3, y_m, t)\} \]

which implies that

\[ S(y_2, y_3, y_m, kt) \geq S(y_1, y_2, y_m, t) \text{ or } S(y_1, y_3, y_m, t). \]

Proceeding in the same way, we get

\[ S(y_n, y_{n+1}, y_m, kt) \geq S(y_{n-1}, y_n, y_m, t) \text{ or } S(y_{n-1}, y_{n+1}, y_m, t) \]

\[ \geq \min \{S(y_{n-2}, y_{n-1}, y_m, t/k), S(y_{n-2}, y_{n+1}, y_m, t/k)\} \]

\[ \geq \ldots \]

\[ \geq \min \{S(y_0, y_1, y_m, t/k^{n-1}), S(y_0, y_{n+1}, y_m, t/k^{n-1})\}. \]
i.e. \( S(y_0, y_{n+1}, y_m, t) \geq S(y_0, y_1, y_m, t/k^n) \) or \( S(y_0, y_{n+1}, y_m, t/k^n) \)

**Case I**

When \( S(y_0, y_{n+1}, y_m, t) \geq S(y_0, y_1, y_m, t/k^n) \).

Then for \( p, q \in \mathbb{N} \) and \( t > 0 \), we have

\[
S(y_n, y_{n+p}, y_{n+p+q}, 3t) \geq S(y_0, y_1, y_{n+p+q}, t/k^n) \cdot S(y_0, y_1, y_{n+p}, t/k^n) \cdot S(y_0, y_{n+2}, y_{n+p+q}, 3t) \cdot S(y_0, y_{n+2}, y_{n+p}, t/k^n) \cdot \ldots \cdot S(y_0, y_{n+p-1}, y_{n+p+q}, t/k^n) \cdot S(y_0, y_{n+p-1}, y_{n+p}, t/k^n)
\]

Taking limit as \( n \to \infty \), we have

\[
\lim_{n \to \infty} S(y_n, y_{n+p}, y_{n+p+q}, 3t) = 1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \text{ (2p-1 times )}
\]

which implies that

\[
S(y_n, y_{n+p}, y_{n+p+q}, 3t) \to 1 \text{ as } n \to \infty.
\]

**Case II**

When \( S(y_0, y_{n+1}, y_m, t) \geq S(y_0, y_{n+1}, y_m, t/k^n) \).

Then on the lines of case I, we have

\[
S(y_0, y_{n+p}, y_{n+p+q}, 3t) \geq S(y_0, y_{n+1}, y_{n+p+q}, t/k^n) \cdot S(y_0, y_{n+2}, y_{n+p+q}, t/k^n) \cdot \ldots \cdot S(y_0, y_{n+p-2}, y_{n+p+q}, t/k^n) \cdot S(y_0, y_{n+p-2}, y_{n+p}, t/k^n) \cdot S(y_0, y_{n+p-1}, y_{n+p+q}, t/k^n) \cdot S(y_0, y_{n+p-1}, y_{n+p}, t/k^n)
\]

Taking limit as \( n \to \infty \), we have

\[
\lim_{n \to \infty} S(y_n, y_{n+p}, y_{n+p+q}, 3t) = 1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \text{ (2p-1 times )}
\]

which implies that
\[ S(y_n, y_{n+p}, y_{n+p+q}, 3t) \to 1 \text{ as } n \to \infty. \]

Thus in both cases, \( \{y_n\} \) is a Cauchy sequence. By the completeness of \( X \), sequence \( \{y_n\} \) and its subsequences \( \{Ax_{2n}\}, \{Bx_{2n-1}\}, \{Px_{2n}\} \) and \( \{Tx_{2n+1}\} \) converge to some \( u \) in \( X \).

Now if we suppose that \( P \) is continuous then \( PAX_{2n}, PPX_{2n} \to Pu \).

Since \((A,P)\) are \( S \)-weakly commuting, therefore
\[ S(APX_{2n}, PAX_{2n}, APX_{2n}, t) \geq S(Ax_{2n}, Px_{2n}, Ax_{2n}, t). \]

On letting \( n \to \infty \), we have
\[ S(\lim_{n \to \infty} APX_{2n}, Pu, \lim_{n \to \infty} APX_{2n}, t) \geq S(u, u, u, t) = 1 \]
which implies that \( APX_{2n} \to Pu \). Now using (iv), we have
\[ S(APX_{2n}, Bx_{2n+1}, u, kt) \geq \min \{ S(PPX_{2n}, Tx_{2n+1}, u, t), S(APX_{2n}, Tx_{2n+1}, u, t), S(Bx_{2n+1}, PPX_{2n}, u, t) \}. \]

On letting \( n \to \infty \), we have
\[ S(Pu, u, u, kt) \geq \min \{ S(Pu, u, u, t), S(Pu, u, u, t), S(u, Pu, u, t) \} \]
or\[ S(Pu, u, u, kt) \geq S(Pu, u, u, t) \]
which implies that \( Pu = u \).

Further using (iv), we have
\[ S(Au, Bx_{2n+1}, u, kt) \geq \min \{ S(Pu, Tx_{2n+1}, u, t), S(Au, Tx_{2n+1}, u, t), S(Bx_{2n+1}, Pu, u, t) \} \]
on letting \( n \to \infty \), we have
\[ S(Au, u, u, kt) \geq \min \{ S(u, u, u, t), S(Au, u, u, t), S(u, u, u, t) \} \]
or\[ S(Au, u, u, kt) \geq S(Au, u, u, t) \]
which implies that \( Au = u \).

Since \( A(X) \subseteq T(X) \), there exists \( v \in X \) such that \( u = Tv = Pu \).

Using (iv), we have
\[ S(u, Bv, u, kt) = S(Au, Bv, u, kt) \]
\[ \geq \min \{ S(Pu, Tv, u, t), S(Au, Tv, u, t), S(Bv, Pu, u, t) \} \]
\[ = \min \{ S(u, u, u, t), S(u, u, u, t), S(Bv, u, u, t) \} \]
or\[ S(u, Bv, u, kt) \geq S(u, Bv, u, t) \]
which implies that \( Bv = u \). Thus \( u = Bv = Tv \). Since \( (T, B) \) are \( S \)-weakly commuting, therefore
\[ S(TBv, BTv, Tbv, t) \geq S(Tv, Bv, Tv, t) = 1 \]
which implies that \( TBv = BTv \) and so \( Tu = Bu \).

Using (iv), we have
\[ S(u, Tu, u, kt) = S(Au, Bu, u, kt) \]
\[ \geq \min \{ S(Pu, Tu, u, t), S(Au, Tu, u, t), S(Bu, Pu, u, t) \} \]
\[ = \min \{ S(u, Tu, u, t), S(u, Tu, u, t), S(Tu, u, u, t) \} \]
\[ S(u, Tu, u, kt) \geq S(u, Tu, u, t) \]
which implies that $u = Tu = Bu$. Hence $u = Tu = Bu = Au = Pu$.
Show $u$ is a common fixed point of $A$, $B$, $P$ and $T$.
Now to prove uniqueness of $u$, let $w$ be another common fixed point of $A$, $B$, $P$ and $T$. Then from (iv), we have

$$S(u, w, u, kt) = S(Au, Bw, u, kt) \geq \min\{S(Pu, Tw, u, t), S(Au, Tw, u, t), S(Bw, Pu, u, t)\}$$

$$= \min\{S(u, w, u, t), S(u, w, u, t), S(w, u, u, t)\}$$

or $S(u, w, u, kt) \geq S(u, w, u, t)$

which implies that $u = w$.
Hence $u$ is a unique common fixed point of $A$, $B$, $P$ and $T$.

To prove our next Theorem for $S$-compatible maps we shall make use of following proposition.

**Proposition 3.2.** Let $A$ and $B$ be $S$-compatible self-mappings of a $S$-fuzzy metric space $X$.
If $Ay = By$ then $ABy = BAy$.

**Proof:** Let $Ay = By$ and $\{x_n\}$ be a sequence in $X$, such that $x_n = y$ for all $n$.
Then $Ax_n, Bx_n \rightarrow Ay$.
Now by the $S$-compatibility of $A$ and $B$, we have $S(ABx_n, BAy, ABy, t) = S(ABx_n, BAx_n, ABx_n, t) \rightarrow 1$, as $n \rightarrow \infty$,
which yields $ABy = BAy$.

**Now we prove following theorem for $S$-compatible maps.**

**Theorem 3.3.** Let $A$, $B$, $P$ and $T$ be self maps of a complete $S$-fuzzy metric space $(X, S, \ast)$ with $t$-norm $\ast$ defined by $a \ast b = \min\{a, b\}, a, b \in [0, 1]$ satisfying

(i) $A(X) \subseteq T(X), B(X) \subseteq P(X)$,

(ii) One of $A$, $B$, $P$ or $T$ is continuous,

(iii) $(A, P)$ and $(B, T)$ are $S$-compatible pairs of maps,

(iv) for all $x, y, z \in X$, $0 < k < 1, t > 0$

$$S(Ax, By, z, kt) \geq \min\{S(Px, Ty, z, t), S(Ax, Ty, z, t), S(By, Px, z, t), S(Ax, Px, z, t), S(By, Ty, z, t)\}$$

or

$$S(x, y, z, t) \rightarrow 1 \text{ as } t \rightarrow \infty.$$}

Then $A$, $B$, $P$ and $T$ have a unique common fixed point in $X$.

**Proof:** Let $x_0 \in X$ be arbitrary. Construct a sequence $\{y_n\}$ in $X$ such that

$y_{2n+1} = Ax_{2n}$ and $y_{2n} = Px_{2n} = Bx_{2n-1}$; $n = 0, 1, 2, \ldots$.

Using (iv), we have

$$S(y_1, y_2, y_m, kt) = S(Ax_0, Bx_1, y_m, kt) \geq \min\{S(Px_0, Tx_1, y_m, t), S(Ax_0, Tx_1, y_m, t), S(By_0, Px_0, y_m, t), S(Ax_0, Px_0, y_m, t), S(By_0, Ty_1, y_m, t)\}$$

$$= \min\{S(y_0, y_1, y_m, t), S(y_1, y_1, y_m, t), S(y_2, y_0, y_m, t), S(y_1, y_0, y_m, t), S(y_2, y_1, y_m, t)\}$$

which implies that $u = w$.

Hence $u$ is a unique common fixed point of $A$, $B$, $P$ and $T$. 
\[ S(y_0, y_1, y_m, t) \leq \min \{ S(y_0, y_1, y_m, t), S(y_1, y_2, y_m, t), S(y_0, y_2, y_m, t) \} \]

which implies that
\[ S(y_1, y_2, y_m, kt) \geq S(y_0, y_1, y_m, t) \text{ or } S(y_0, y_2, y_m, t). \]

Further using (iv), we have
\[ S(y_2, y_3, y_m, kt) = S(Bx_1, Ax_2, y_m, kt) \]
\[ \geq \min \{ S(Px_2, Tx_1, y_m, t), S(Ax_2, Tx_1, y_m, t), S(Bx_1, Px_2, y_m, t), S(Ax_2, Px_2, y_m, t), S(Bx_1, Tx_1, y_m, t) \} \]
\[ = \min \{ S(y_2, y_1, y_m, t), S(y_3, y_1, y_m, t), S(y_2, y_2, y_m, t), S(y_3, y_2, y_m, t), S(y_2, y_1, y_m, t) \}. \]

which implies that
\[ S(y_2, y_3, y_m, kt) \geq S(y_1, y_2, y_m, t) \text{ or } S(y_1, y_3, y_m, t). \]

Again with the similar process as in Theorem 3.1, we can show \( \{y_n\} \) is a Cauchy sequence. By the completeness of \( X \), sequence \( \{y_n\} \) and its subsequences \( \{Ax_{2n}\}, \{Bx_{2n-1}\}, \{Px_{2n}\} \) and \( \{Tx_{2n+1}\} \) converge to some \( u \) in \( X \). Now if we suppose that \( P \) is continuous then \( PAx_{2n} \to Pu \).

Since \( (A,P) \) are S-compatible, therefore
\[ \lim_{n \to \infty} S(PAx_{2n}, APx_{2n}, PAx_{2n}, t) = 1, \]
where \( \{x_n\} \) is a sequence such that
\[ \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Px_{2n} = u. \]

Thus, we have \( S(Pu, \lim APx_{2n}, Pu, t) = 1 \)

which implies that \( \lim APx_{2n} = Pu. \)

Now using (iv), we have
\[ S(APx_{2n}, Bx_{2n+1}, u, kt) \geq \min \{ S(PPx_{2n}, Tx_{2n+1}, u, t), S(APx_{2n}, Tx_{2n+1}, u, t), S(Bx_{2n+1}, PPx_{2n}, u, t), S(APx_{2n}, PPx_{2n}, u, t), S(Bx_{2n+1}, Tx_{2n+1}, u, t) \}. \]

On letting \( n \to \infty \), we have
\[ S(Pu, u, u, kt) \geq \min \{ S(Pu, u, u, t), S(Pu, u, u, t), S(u, Pu, u, t), S(Pu, Pu, u, t), S(u, u, u, t) \} \]
\[ = S(Pu, u, u, t) \]

which implies that
\[ S(Pu, u, u, kt) \geq S(Pu, u, u, t). \]

Hence \( Pu = u. \)

Further using (iv), we have
\[ S(Au, Bx_{2n+1}, u, kt) \geq \min \{ S(Pu, Tx_{2n+1}, u, t), S(Au, Tx_{2n+1}, u, t), S(Bx_{2n+1}, Pu, u, t), S(Au, Pu, u, t), S(Bx_{2n+1}, Tx_{2n+1}, u, t) \}. \]

On letting \( n \to \infty \), we have
Common fixed point theorems

\[ S(Au, u, u, kt) \geq \min \{ S(u, u, u, t), S(Au, u, u, t), S(u, u, u, t), S(Au, u, u, t) \} \]

this implies that
\[ S(Au, u, u, kt) \geq S(Au, u, u, t) \]

Hence \( Au = u \). Since \( A(X) \subseteq T(X) \), there exists \( v \in X \) such that \( u = Tv = Pu \).

Using (iv), we have
\[ S(u, Bv, u, kt) = S(Au, Bv, u, kt) \geq \min \{ S(Pu, Tv, u, t), S(Au, Tv, u, t), S(Bv, Pu, u, t), S(Au, Pu, u, t), S(Bv, Tv, u, t) \} \]

This implies that
\[ S(u, Bv, u, kt) \geq S(u, Bv, u, t) \]

which implies that \( Bv = u \). Thus \( u = Bv = Tv \).

By compatibility of \( (T, B) \) and from proposition 3.2, we have \( TBv = BTv \) and so \( Tu = Bu \).

Using (iv), we have
\[ S(u, Tu, u, kt) = S(Au, Bu, u, kt) \geq \min \{ S(Pu, Tu, u, t), S(Au, Tu, u, t), S(Bu, Pu, u, t), S(Au, Pu, u, t), S(Bu, Tu, u, t) \} \]

which implies that \( Tu = Tu = Bu \). Hence \( u = Tu = Bu = Au = Pu \), shows \( u \) is a common fixed point of \( A, B, P \) and \( T \).

Now to prove uniqueness of \( u \), let \( w \) be another common fixed point of \( A, B, P \) and \( T \).

Then from (iv), we have
\[ S(u, w, u, kt) = S(Au, Bw, u, kt) \geq \min \{ S(Pu, Tw, u, t), S(Au, Tw, u, t), S(Bw, Pu, u, t), S(Au, Pu, u, t), S(Bw, Tw, u, t) \} \]

This implies that
\[ S(u, w, u, kt) \geq S(u, w, u, t) \]

Hence \( u = w \). Thus \( u \) is a unique common fixed point of \( A, B, P \) and \( T \).

4. Common fixed point theorems for RS-weakly commuting maps in complete S-fuzzy metric spaces

Pant, R.P. [5] introduced the notion of R-weakly commutativity of mappings in metric spaces as follows.
Definition 4.1. Two mappings \( f \) and \( g \) of a metric space \( (X, d) \) into itself are said to be \( R \)-weakly commuting, provided there exists some positive real number \( R \) such that 
\[
d(fgx, gfx) \leq Rd(fx, gx)
\]
for each \( x \) in \( X \).
Later on Vasuki, R. \[6\] defined \( R \)-weakly commuting maps in fuzzy metric spaces as follows.

**Definition 4.2**: The mappings \( f \) and \( g \) of a fuzzy metric space \( (X, M, *) \) into itself are \( R \)-weakly commuting, provided there exists some positive real number \( R \) such that
\[
M(fgx, gfx, t) \geq M(fx, gx, t/R)
\]
for all \( x \) in \( X \).

**Now we define RS-weakly commuting maps in S-fuzzy metric space \( (X, S, *) \)**

**Definition 4.3.** Two self maps \( A \) and \( B \) of a \( S \)-fuzzy metric space \( (X, S, *) \) are said to be \( RS \)-weakly commuting, if there exists some positive real number \( R \) such that 
\[
S(ABx, BAx, y, t) \geq S(Ax, Bx, z, t/R)
\]
where \( y = ABx \) or \( BAx \) and \( z = Ax \) or \( Bx \) for all \( x \in X \).

Obviously, \( S \)-weak commutativity implies \( RS \)-weak commutativity. But convers is not true always.

This can be seen in following example.

**Example 4.1** - Let \( X \) be the set of real numbers. Define \( S(x, y, z, t) = \min\{M(x, y, t), M(y, z, t), M(z, x, t)\} \)
\[
t + d(x, y)
\]
also define self maps \( A \) and \( B \) of \( X \) by \( Ax = 2x - 1 \), \( Bx = x^2 \).
Then \( AB \neq BA \) and clearly
\[
S(ABx, BAx, ABx, t) = S(Ax, Bx, Ax, t/R)
\]
for \( R = 2 \) and so
\[
S(ABx, BAx, ABx, t) \geq S(Ax, Bx, Ax, t/R)
\]
is true.
Also it can easily be seen that
\[
S(ABx, BAx, ABx, t) \geq S(Ax, Bx, Ax, t)
\]
is not true.

Vasuki, R. \[6\] proved the following.

**Theorem 4.1.** Let \( (X, M, *) \) be a complete fuzzy metric space and let \( f \) and \( g \) be \( R \)-weakly commuting self mappings of \( X \) satisfying the conditions
\[
M(fx, fy, t) \geq r(M(gx, gy, t))
\]
where \( r : [0, 1] \rightarrow [0, 1] \) is a continuous function such that \( r(t) > t \) for each \( 0 < t < 1 \). The sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) are such that 
\[
x_n \rightarrow x, \quad y_n \rightarrow y, \quad t > 0 \implies M(x_n, y_n, t) \rightarrow M(x, y, t)
\]
If the range of \( g \) contains the range of \( f \) and if either \( f \) or \( g \) is continuous, then \( f \) and \( g \) have a unique common fixed point.

**Now we prove the following theorem**

**Theorem 4.2.** Let \( (X, S, *) \) be a complete \( S \)-fuzzy metric space and \( A \) and \( B \) are \( RS \)-weakly commuting self maps of \( X \) satisfying
(i) $A(X) \subset B(X)$,
(ii) $A$ or $B$ is continuous,
(iii) $S(Ax, Ay, Az, t) \geq \phi \{ S(Bx, By, Bz, t) \}$

where $\phi : [0, 1] \rightarrow [0, 1]$ is a continuous function such that $\phi(t) > t$ for each $0 < t < 1$ and $\phi(1) = 1$.

Then $A$ and $B$ have a unique common fixed point.

**Proof**: Let $x_0$ be an arbitrary point in $X$. Since $A(X) \subset B(X)$ choose a point $x_1$ in $X$ such that $Ax_0 = Bx_1$. In general choose $x_{n+1}$ such that $Ax_n = Bx_{n+1}$. Then for $t > 0$

$$S(Ax_n, Ax_{n+1}, A x_{n+p}, t) \geq \phi \{ S(Bx_n, Bx_{n+1}, Bx_{n+p}, t) \} = \phi \{ S(Ax_{n-1}, Ax_n, A x_{n+p-1}, t) \}$$

or $S(Ax_n, Ax_{n+1}, A x_{n+p}, t) > S(Ax_{n-1}, Ax_n, A x_{n+p-1}, t)$, since $\phi(t) > t$ (1)

Thus $\{ S(Ax_n, Ax_{n+1}, A x_{n+p}, t), \ n \geq 0 \}$ is an increasing sequence of positive real numbers in $[0, 1]$ and therefore tends to a limit $L \leq 1$. We claim that $L = 1$. For if $L < 1$, on making $n \rightarrow \infty$ in (1), we have

$L \geq \phi(L) > L$ which is a contradiction. Hence $L = 1$.

Now for $p, q \in \mathbb{N}$ and $t > 0$, we have

$$S(Ax_n, Ax_{n+p}, Ax_{n+p+q}, t) \geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3)*S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+p}, Ax_{n+p+q}, t/3)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3)*S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3) * S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3) * S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3) * S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3) * S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3) * S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq S(Ax_n, Ax_{n+1}, Ax_{n+p+q}, t/3) * S(Ax_n, Ax_{n+1}, Ax_{n+p}, t/3)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p+q}, t/3^2)$$

$$* S(Ax_{n+1}, Ax_{n+2}, Ax_{n+p}, t/3^2)$$

$$\geq \ldots$$

Taking limit $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} S(Ax_n, Ax_{n+p}, Ax_{n+p+q}, t) \geq 1 * 1 * 1 * \ldots * 1 \quad (2p-1 \text{ times})$$

or

$$\lim_{n \rightarrow \infty} S(Ax_n, Ax_{n+p}, Ax_{n+p+q}, t) = 1$$

Thus $\{Ax_n\}$ is a Cauchy sequence. By the completeness of the space, there is a point $z$ in $X$, such that $\lim \ Ax_n = z$.

$$n \rightarrow \infty$$
Now let the mapping $A$ be continuous. Then $AAX_n \to Az$ and $ABx_n \to Az$. Since $A$ and $B$ are RS-weakly commuting, we have

$$S(ABx_n, BAX_n, ABx_n, t) \geq S(Ax_n, Bx_n, Ax_n, t/R).$$

On letting $n \to \infty$, we have

$$S(Az, \lim BAX_n, Az, t) \geq S(z, z, z, t/R) = 1.$$

This implies that $\lim BAX_n = Az$. 

Suppose $Az \neq z$, then 

$$S(z, Az, z, t) = \lim S(Ax_n, AAX_n, Ax_n, t)$$

\[ \geq \lim_{n \to \infty} \phi \{S(Bx_n, BAX_n, Bx_n, t)\} \text{ from (iii)} \]

\[ = \phi \lim_{n \to \infty} \{S(Bx_n, BAX_n, Bx_n, t)\} \]

\[ = \phi \{S(z, Az, z, t)\} > S(z, Az, z, t), \quad \text{since } \phi(t) > t \]

i.e. $S(z, Az, z, t) > S(z, Az, z, t)$

which implies that $Az = z$. Again $A(X) \subset B(X)$, we can find $z_1$ in $X$ such that $z = Az = Az_1$.

Now using (iii), we have

$$S(AAX_n, Az_1, AAX_n, t) \geq \phi \{S(Bz_1, Bz_1, Bz_1, t)\}.$$

On letting $n \to \infty$, we get

$$S(Az, Az_1, Az, t) \geq \phi \{S(Az, z, Az, t)\} = 1, \quad \text{since } \phi(1) = 1 \text{ and } Az = z$$

which implies that $Az = Az_1$ i.e. $z = Az = Az_1 = Bz_1$.

Also for any $t > 0$

$$S(Az, Bz, Az, t) = S(ABz_1, BAz_1, ABz_1, t) \geq S(Az_1, Bz_1, Az_1, t/R),$$

by RS-weakly commutativity

$$= 1$$

This implies that $Az = Bz$.

Thus $z$ is a common fixed point of $A$ and $B$.

To prove the uniqueness, let $z' \neq z$ be another common fixed point of $A$ and $B$. Now

$$S(z, z', z, t) = S(Az, Az', Az, t)$$

\[ \geq \phi \{S(Bz, Bz', Bz, t)\} \]

\[ = \phi \{S(z, z', z, t)\} \]

or $S(z, z', z, t) > S(z, z', z, t)$, since $\phi(t) > t$

a contradiction. Thus $z = z'$.

Hence $A$ and $B$ have a unique common fixed point.
Common fixed point theorems

References


M. S. Rathore, Government P.G. College Sehore M.P., India

Deepak Singh, Corporate Institute Of Science & Technology, Bhopal-462021, M.P., India

Naval Singh, Govt. Science and Commerce College, Benazir Bhopal-462021, M.P., India

Received: April, 2009