Soliton and Numerical Solutions of
the Burgers’ Equation and Comparing them

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Abstract

One-dimensional Burgers’ equation is a nonlinear partial differential equation that is a simple form of Navier-stocks equations. Burgers’ equation is known as a model equation, and that is why it is important. This equation has different types and each of them has special applications. Burgers’ equation and their solutions have been discussed in this paper. Analytical solution of this equation includes soliton solution which introduces two basic elements of solution and their application. Their application is used in Burgers’ equation, and the soliton’s solution is obtained from this equation. Numerical methods used in this paper include 8-point and 12-point numerical methods which are multisymplectic. In this paper, by considering different values of soliton’s solution parameters, a comparison is made between 8-point and 12-point numerical methods. The error results are calculated through Norm2 and Norm infinity which are shown in the tables. Moreover, the error curves are shown in the tables as well.

1 Introduction

Partial differential equations (PDEs) arise in many fields of science, particularly in physics, engineering, chemistry and finance which are fundamental for the mathematical formulation of continuum models[10]. The study of PDEs was started in the 18th century in the work of Euler, d’Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally as the principal mode of analytical study of models in the physical science. Beginning in the middle of the 19th century, particularly with the work of Riemann, PDEs also became an essential tool in other branches of mathematics[4]. One of the most important PDEs in the theory of nonlinear conservation laws, is Burgers’ equation. This equation has a large variety of applications in the modeling of water in unsaturated soil, dynamics of soil water, statistics of flow problems, mixing and turbulent diffusion,
cosmology and seismology[7,12,11]. The Burgers’ equation is a nonlinear equation, very similar to the Navier-stokes equation which could serve as a nonlinear analog of the Navier-stokes equations. This single equation has a convection term, a diffusion term and a time-dependent term:

\[ u_t + uu_x = \nu u_{xx}. \]  

Burgers’ equation is parabolic when the viscous term is included. If the viscous term is neglected, the remaining equation is hyperbolic. If the viscous term is dropped from the Burgers’ equation, the nonlinearity allows discontinuous solutions to develop. In Burgers’ equation discontinuities may appear in finite time, even if the initial condition is smooth. They give rise to the phenomenon of shock waves with important applications in physics [4]. These properties make Burgers’ equation a proper model for testing numerical algorithms in flows where severe gradients or shocks are anticipated[8].

Many soliton solutions of nonlinear partial differential equations can be written as a polynomial in two elementary functions which satisfy a projective (hence linearizable) Riccati system. From that property, we deduce a method for building these solutions by determining only a finite number of coefficients. This method is much shorter and obtains more solutions than the one which consists of summing a perturbation series built from exponential solutions of the linearized equation. In this paper, this method is used obtained solution of the Burgers’ equation[3].

Discretization methods are well-known techniques for solving Burgers’ equation. Ascher and McLachlan established a method based on box scheme [1]. The name of “box scheme” is a general name for several numerical schemes of different origins [5]. The box scheme, a compact scheme in both x and t centered at cell(box), has been used for many years [2]. For example Keller [6] introduced this scheme on the 1-D heat equation in the 1970s. Box schemes types are conservative schemes, i.e. schemes which guarantee, for equations in divergence form, an exact conservation of the flux at the level of the box. The first one has been introduced in the 1980’s in compressible computational fluid dynamics. As in Keller’s scheme, the basic idea is that locating the degrees of freedom at the center of the faces instead at the center of the cells, could be more interesting for an accurate evaluation of conservative fluxes. In this paper whose inspiration is [2], we consider some multisymplectic box methods for the Burgers’ equation[8].

The paper is organized as follows: In section 2, the soliton solution is used to obtain the analytical solutions of the Burgers’ equation. In section 3, the multisymplectic box scheme is used to obtain the numerical solutions of the Burgers’ equation. In section 4, numerical experiments are performed to test the accuracy and efficiency of the numerical solutions compared with the soliton solutions.
2 soliton solutions by using projective matrix Riccati Equation

To illustrate the basic concepts of the projective matrix Riccati system method, we consider a given PDE in two independent variables given by

\[ F(u, u_x, u_t, u_{xx}, ...) = 0. \]  

(2)

We first consider its soliton solutions \( u(x, t) = u(\xi), \xi = x + ct \) or \( \xi = x - ct \); then Eq. (2) becomes an ordinary differential equation. The method is explained briefly as we will use a similar procedure to obtain soliton solution of the Burgers’ equation by employing coupled Riccati equation of projective type. In this method, as discussed in [3], we use two functions \( \sigma \) (bell shaped) and \( \tau \) (kink-shaped) which are

\[
\sigma(\theta) = \frac{b}{\cosh(\theta) + \mu}, \quad \tau(\theta) = \frac{\sinh(\theta)}{\cosh(\theta) + \mu},
\]  

(3)

where \( \mu, b \) are constants and \( \theta \) is the independent variable which we take it to be a function of \( \xi = x - ct \). These functions represent the general two-parameter solution of the coupled system of projective Riccati equations

\[
\dot{\sigma} = -\sigma \tau, \quad \dot{\tau} = -\tau^2 - \frac{\mu}{b} \sigma + 1,
\]  

(4)

which admits the first integral

\[
\left[ \frac{1}{\sigma} - \frac{\mu}{b} \right]^2 - \frac{\tau^2}{\sigma^2} = b^{-2},
\]  

(5)

and variable \( \theta \) is complex, so that the two functions can be trigonometric or hyperbolic. Both \( \sigma \) and \( \tau \) have simple movable poles (i.e. whose location depends on integration constants) in the \( \theta \) complex plane, except for \( \mu = \pm 1 \), in which case \( \sigma \) has double poles. In order to prevent this change in the pole order of \( \sigma \), we choose \( b = \sqrt{\mu^2 - 1} \) and consider the system

\[
\dot{\sigma} = -\sigma \tau, \quad \dot{\tau} = -\tau^2 - \mu_0 \sigma + 1 \text{ modulo } 1 - \tau^2 - 2\mu_0 \sigma + \sigma^2 = 0,
\]  

(6)

where;

\[
\mu_0 = \frac{\mu}{\sqrt{\mu^2 - 1}} \text{ with the restriction } \mu_0^2 \neq 1.
\]  

(7)

The class of equations to which the method applies is made of the nonlinear PDE \( E(u) = 0 \), polynomial in \( u \) and its derivatives.
The first step consists of determining degree \( p \) in the solution \( u \) in \( (\sigma, \tau) \), which must be a positive integer. Again at this stage, some transformation \( u \mapsto u^\alpha \) may take place in order to satisfy this requirement.

In the second step, one defines \( u \) as the most general polynomial in \( \sigma, \tau \) with a global degree \( p \) in \( (\sigma, \tau) \) and a degree one in \( \tau \) by using the first integral (5).

\[
u = \sum_{l=0}^{1} \sum_{j=0}^{p-l} c_{j,l} \sigma^j \tau^l(\theta) \quad (c_{p,0}, c_{p-1,1}) \neq (0, 0) \tag{8}\]

Then, one puts the LHS \( E(u) \) under the same canonical form by eliminating any derivatives of \( (\sigma, \tau) \) and any power of \( \tau \) higher than one with the definition of the projective Riccati system (6);

\[
E(u) = \sum_{l=0}^{1} \sum_{j=0}^{Q-l} E_{j,l} \sigma^j \tau^l. \tag{9}\]

The next step consists of solving the set of \( 2Q+1 \) determining equations;

\[
\forall j, l : \quad E_{j,l}(\mu_0, \dot{\theta}, c, c_{m,n}) = 0, \quad (\mu_0^2 - 1)\theta' \neq 0, \tag{10}\]

for the \( 2p + 4 \) unknowns \( \mu_0, \dot{\theta}, c, c_{m,n} \), in a way which avoids finding several times the same solution under different representations. To illustrate the procedure, the 1D Burgers’ equation is given in the following.

### 2.1 Soliton solution in one-dimensional Burgers’ equation

Let us consider the 1D Burgers’ equation which has the form

\[
\alpha uu_x - \nu u_{xx} = 0, \tag{11}\]

where \( \alpha \) and \( \nu \) are arbitrary constants. In order to solve Eq. (11) by the above procedure, we use the wave transformation \( u(x, t) = U(\xi) \) with wave variable \( \xi = x - ct; \) Eq. (11) takes on the form of an ordinary differential equation as follows:

\[
-cU'' + \frac{\alpha}{2}(U^2)' - \nu U'' = 0. \tag{12}\]

Integrating Eq. (12) once with respect to \( \xi \) and setting the constant of integration to zero, we obtain

\[
-cU + \frac{\alpha}{2}(U^2) - \nu U' = 0. \tag{13}\]

Balancing the order of \( U^2 \) with the order of \( U' \) in Eq. (13), we find \( p = 1 \). So the solution takes on the form

\[
U(\xi) = d_0 + d_1 \tau(\theta) + d_2 \sigma(\theta). \tag{14}\]
Inserting $\theta$ equal $k\xi$ and inserting Eq.(14) into Eq.(13) and making use of Eq.(6), using the Maple package, we obtain a system of algebraic equations for $d_0, d_1, d_2, \mu_0$ and $k$, of the following form:

\[-cd_0 + \frac{1}{2} \alpha d_0^2 + \frac{1}{2} \alpha d_2^2 = 0,\]
\[-cd_2 + \alpha d_0 d_2 = 0,\]
\[-cd_1 + \alpha d_0 d_1 - \alpha d_2^2 \mu_0 - \nu d_2 k \mu_0 = 0,\]
\[\frac{1}{2} \alpha d_1^2 + \frac{1}{2} \alpha d_2^2 + \nu d_2 k = 0,\]
\[\alpha d_1 d_2 + \nu d_1 k = 0.\]

These equations give the following two cases: It’s noticeable to say that here we have considered $\theta$ equal $k\xi$.

Case(1): $d_0 = \frac{c}{\alpha}$, $d_1 = \frac{\nu k}{\alpha}$, $d_2 = -\frac{\nu k}{\alpha}$, $\mu_0 = \mu_0$; the soliton solution is given by

\[u(x, t) = \frac{c}{\alpha} + \frac{\nu k b}{\alpha(\cosh(k(x - ct)) + \mu)} - \frac{\nu k \sinh(k(x - ct))}{\alpha(\cosh(k(x - ct)) + \mu)}.\] (15)

Case(2): $d_0 = \frac{c}{\alpha}$, $d_1 = -\frac{\nu k}{\alpha}$, $d_2 = -\frac{\nu k}{\alpha}$, $\mu_0 = \mu_0$; the soliton solution is given by

\[u(x, t) = \frac{c}{\alpha} - \frac{\nu k b}{\alpha(\cosh(k(x - ct)) + \mu)} - \frac{\nu k \sinh(k(x - ct))}{\alpha(\cosh(k(x - ct)) + \mu)}.\] (16)

3 Box method

As we mentioned in the introduction, the box scheme has been proposed basically for many years. This type of compact discretization method in both $x$ and $t$ is centered at a cell(box), whose corners are $(x_i, t_n)$, $(x_{i+1}, t_n)$, $(x_i, t_{n+1})$, $(x_{i+1}, t_{n+1})$.
Based on this compact scheme some multisymplectic box and fully implicit narrow box scheme can be constructed. Only the first one is considered on Burgers’ equation here.

3.1 Multisymplectic box scheme

Implementing the box scheme on Hamiltonian PDEs like KdV equation is proposed in [2]. Zhao and Qin[9] proved that the following 12-point scheme which can be
represented by its computational stencil, has a multisymplectic result for the KdV equation:

\[
\frac{1}{16\Delta t} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 \end{pmatrix} u = \frac{1}{4\Delta x} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \dot{V} \left( \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \right) + \frac{\nu}{4(\Delta x)^3} \begin{pmatrix} -1 & 3 & -3 & 1 \\ -2 & 6 & -6 & 2 \\ -1 & 3 & -3 & 1 \end{pmatrix} u.
\]

(17)

Recall that the KdV equation is given by

\[
u_t = \alpha(u^2)_x + \rho u_x + \nu u_{xxx} = V'(u)_x + \nu u_{xxx}, \quad V(u) = \frac{\alpha}{3} u^3 + \frac{\rho}{2} u^2.
\]

(18)

It is assumed that this equation has initial conditions \(u(x,0)\) and periodic boundary conditions (in time). The above discretization method which is obtained from the following 8-point scheme by averaging at time levels \(n\) and \(n+1\), is stable [2].

\[
\frac{1}{8\Delta t} \begin{pmatrix} 1 & 3 & 3 & 1 \\ -1 & -3 & -3 & -1 \end{pmatrix} u = \frac{1}{2\Delta x} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \dot{V} \left( \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \right) + \frac{\nu}{2(\Delta x)^3} \begin{pmatrix} -1 & 3 & -3 & 1 \\ -1 & 3 & -3 & 1 \end{pmatrix} u.
\]

(19)

### 3.2 Application in Burgers’ equation

In this section, we apply two 12-point and 8-point multisymplectic schemes on Burgers’ equation. For applying these discretization methods on Burgers’ equation, Burgers’ equation can be represented by the following formula[8]

\[
u_t = -uu_x + \nu u_{xx} = -V'(u) + \nu u_{xx}, \quad V(u) = \frac{u^3}{6}.
\]

(20)

By applying the 8-point multisymplectic schemes on Burgers’ equation, we have the following:

\[
u_t = \frac{1}{8\Delta t} \begin{pmatrix} 1 & 3 & 3 & 1 \\ -1 & -3 & -3 & -1 \end{pmatrix} u = \frac{1}{2\Delta x} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \dot{V} \left( \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \right) + \frac{1}{2(\Delta x)^2} \begin{pmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \end{pmatrix} u.
\]

(21)

Also by applying the 12-point multisymplectic scheme on Burgers’ equation, the following discretization method with stencil notation is resulted:

\[
u_t = \frac{1}{16\Delta t} \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -3 & -3 & -1 \end{pmatrix} u = -\frac{1}{4\Delta x} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \dot{V} \left( \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u \right) + \frac{1}{4(\Delta x)^2} \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix} u.
\]

(22)

As we saw already in constructing the two above schemes, when using the 8-point scheme for initializing 12-point scheme, identical results up to round off error level are obtained[8].
4 Numerical results

To show the efficiency of the present method for numerical solutions in comparison with the soliton solutions, we report the computed errors of the solution defined by

\[ \|E\|_\infty = \max_{1 < i < n_{ip}} |u_i - u_{is}|, \]  

and

\[ \|E\|_2 = \sqrt{\sum_{i=1}^{n_{ip}} (u_i - u_{is})^2}. \]  

where \( n_{ip} \) is the number of interior points, also \( u_{is} \) and \( u_i \) are the exact and computed values of the solution \( u \) at point \( i \).

4.1 Comparison between soliton and numerical solutions

In section 2, two kinds of soliton solutions were achieved for Burgers’ equation which are arranged in the wording of:

(The first solution)

\[ u(x, t) = c + \frac{\nu kb}{\alpha(cosh(k(x - ct)) + \mu)} - \frac{\nu k sinh(k(x - ct))}{\alpha(cosh(k(x - ct)) + \mu)}, \]  

(The second solution)

\[ u(x, t) = c - \frac{\nu kb}{\alpha(cosh(k(x - ct)) + \mu)} - \frac{\nu k sinh(k(x - ct))}{\alpha(cosh(k(x - ct)) + \mu)}. \]  

In both equations we have \( b = \sqrt{\mu^2 - 1} \) and \( c^2 = (\nu k)^2 \) under the condition that \( (\frac{\mu}{\sqrt{\mu^2 - 1}})^2 \neq 1 \). Also in section 3, we introduced two methods including numerical multisymplectic 8-point and 12-point from Box method. The purpose of this section is to compare soliton and numerical solutions with 1D Burgers’ equation \( (\alpha = 1) \) under the initial conditions, boundary amounts and present parameters. The initial condition is taken into consideration for the first solution as follows:

\[ u(x, 0) = c + \frac{\nu kb}{\alpha(cosh(kx) + \mu)} - \frac{\nu k sinh(kx)}{\alpha(cosh(kx) + \mu)}. \]  

and for the second solution as the following:

\[ u(x, 0) = c - \frac{\nu kb}{\alpha(cosh(kx) + \mu)} - \frac{\nu k sinh(kx)}{\alpha(cosh(kx) + \mu)}. \]  

Also boundary amounts in both types are \( u(0, t) = u(1, t) = 0 \). All the error values are evaluated in \( t = 0.05 \).
4.2 The outcomes obtained from the comparison of the first solution with the 8-point method

Table 1
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 10^{-5}$, $\nu = 10^{-5}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$6.7064 \times 10^{-6}$</td>
<td>$3.529 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$6.520 \times 10^{-6}$</td>
<td>$3.466 \times 10^{-5}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$7.976 \times 10^{-6}$</td>
<td>$4.054 \times 10^{-5}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$7.856 \times 10^{-6}$</td>
<td>$4.021 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation declines as well.
Table 2
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 10^{-4}$, $\nu = 10^{-4}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$3.516 \times 10^{-5}$</td>
<td>$2.189 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$3.370 \times 10^{-5}$</td>
<td>$2.128 \times 10^{-4}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$4.746 \times 10^{-5}$</td>
<td>$2.700 \times 10^{-4}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$4.646 \times 10^{-5}$</td>
<td>$2.662 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation declines as well.

Table 3
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 75 \times 10^{-5}$, $\nu = 75 \times 10^{-5}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$2.902 \times 10^{-4}$</td>
<td>$1.763 \times 10^{-3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$2.752 \times 10^{-4}$</td>
<td>$1.698 \times 10^{-3}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$3.805 \times 10^{-4}$</td>
<td>$2.136 \times 10^{-3}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$3.734 \times 10^{-4}$</td>
<td>$2.108 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation declines as well.

4.3 The outcomes obtained from the comparison of the first solution with the 12-point method

Table 4
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 10^{-5}$, $\nu = 10^{-5}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$2.364 \times 10^{-6}$</td>
<td>$9.346 \times 10^{-6}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$1.508 \times 10^{-6}$</td>
<td>$5.673 \times 10^{-6}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$2.728 \times 10^{-6}$</td>
<td>$1.036 \times 10^{-5}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$2.708 \times 10^{-6}$</td>
<td>$1.030 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation declines as well.
As it has been shown in the above table, for positive amounts of $\mu$ the deviation inclines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation declines as well.

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$, as $\mu$ declines, deviation inclines.

**4.4 The outcomes obtained from the comparison of the second solution with the 8-point method**

As it has been shown in the above table, for positive amounts of $\mu$ the deviation inclines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation inclines as well.
Table 8
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$, $c = 10^{-4}$, $\nu = 10^{-4}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$4.310 \times 10^{-5}$</td>
<td>$2.527 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$4.379 \times 10^{-5}$</td>
<td>$2.553 \times 10^{-4}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$1.344 \times 10^{-5}$</td>
<td>$1.060 \times 10^{-4}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$2.173 \times 10^{-5}$</td>
<td>$1.588 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation inclines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation inclines as well.
Table 9
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 75 \times 10^{-5}$, $\nu = 75 \times 10^{-5}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$3.494 \times 10^{-4}$</td>
<td>$2.011 \times 10^{-3}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$3.538 \times 10^{-4}$</td>
<td>$2.029 \times 10^{-3}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$1.107 \times 10^{-4}$</td>
<td>$8.836 \times 10^{-4}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$1.865 \times 10^{-4}$</td>
<td>$1.294 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation inclines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation inclines as well.

4.5 The outcomes obtained from the comparison of the second solution with the 12-point method

Table 10
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 10^{-5}$, $\nu = 10^{-5}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$3.972 \times 10^{-6}$</td>
<td>$1.540 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$2.645 \times 10^{-6}$</td>
<td>$9.686 \times 10^{-6}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$2.771 \times 10^{-6}$</td>
<td>$1.068 \times 10^{-5}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$2.604 \times 10^{-6}$</td>
<td>$9.698 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$ when $\mu$ declines, the deviation declines as well.

Table 11
Error results for $\Delta t = 0.01$, $\Delta x = 0.01$, $k = 1$,
$c = 10^{-4}$, $\nu = 10^{-4}$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$1.817 \times 10^{-5}$</td>
<td>$9.883 \times 10^{-5}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$1.784 \times 10^{-5}$</td>
<td>$9.783 \times 10^{-5}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$1.778 \times 10^{-5}$</td>
<td>$9.674 \times 10^{-5}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$1.785 \times 10^{-5}$</td>
<td>$9.786 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

As it has been shown in the above table, for positive amounts of $\mu$ the deviation declines while $\mu$ increases, but for negative amounts of $\mu$, as $\mu$ declines, deviation inclines.
As it has been shown in the above table, for positive amounts of $\mu$ the deviation inclines while $\mu$ increases, but for negative amounts of $\mu$, as $\mu$ declines, deviation declines.

5 conclusions and future work

The comparison has been examined from a more general perspective. The tables presented in this part have been provided for a better comparison for different quantities of $\nu$.

5.1 Error values of the 8-point method for the first solution

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$b = \sqrt{\mu^2 - 1}$</th>
<th>$|E|_\infty$</th>
<th>$|E|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\sqrt{3}$</td>
<td>$1.051 \times 10^{-4}$</td>
<td>$6.037 \times 10^{-4}$</td>
</tr>
<tr>
<td>3</td>
<td>$\sqrt{8}$</td>
<td>$1.052 \times 10^{-4}$</td>
<td>$6.055 \times 10^{-4}$</td>
</tr>
<tr>
<td>-2</td>
<td>$\sqrt{3}$</td>
<td>$1.063 \times 10^{-4}$</td>
<td>$6.094 \times 10^{-4}$</td>
</tr>
<tr>
<td>-3</td>
<td>$\sqrt{8}$</td>
<td>$1.057 \times 10^{-4}$</td>
<td>$6.072 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

5.2 Error values of the 12-point method for the first solution

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\nu = 10^{-5}$</th>
<th>$\nu = 10^{-4}$</th>
<th>$\nu = 75 \times 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.000006706452797</td>
<td>0.00003516447971</td>
<td>0.0002902933166</td>
</tr>
<tr>
<td>3</td>
<td>0.00006520714636</td>
<td>0.00003370726045</td>
<td>0.0002752606338</td>
</tr>
<tr>
<td>-2</td>
<td>0.00007976075602</td>
<td>0.00004746032126</td>
<td>0.0003805788594</td>
</tr>
<tr>
<td>-3</td>
<td>0.00007856464338</td>
<td>0.00004646442456</td>
<td>0.0003734213356</td>
</tr>
<tr>
<td>2</td>
<td>0.00003529500543</td>
<td>0.0002189548894</td>
<td>0.001763248762</td>
</tr>
<tr>
<td>3</td>
<td>0.00003466734140</td>
<td>0.0002128808268</td>
<td>0.001698218700</td>
</tr>
<tr>
<td>-2</td>
<td>0.00004054761330</td>
<td>0.0002700971145</td>
<td>0.002136699037</td>
</tr>
<tr>
<td>-3</td>
<td>0.00004021964101</td>
<td>0.0002662718364</td>
<td>0.002108626871</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|c|}
\hline
\mu & \nu = 10^{-5} & \nu = 10^{-4} & \nu = 75 \times 10^{-5} \\
\hline
2 & 0.00002364542004 & 0.0001802653661 & 0.0001060933166 \\
3 & 0.0001508296782 & 0.0001830726045 & 0.000105606338 \\
-2 & 0.0002728130986 & 0.0001771826283 & 0.0001053788594 \\
-3 & 0.0002708774520 & 0.0001794713858 & 0.0001058213356 \\
\hline
\end{array}
\]

Table 16

2-norm of Errors

\[
\begin{array}{|c|c|c|c|}
\hline
\mu & \nu = 10^{-5} & \nu = 10^{-4} & \nu = 75 \times 10^{-5} \\
\hline
2 & 0.000009346185732 & 0.00009823538454 & 0.00006090220150 \\
3 & 0.00005673270209 & 0.00009946821588 & 0.00006059946181 \\
-2 & 0.0001036236494 & 0.00009649003726 & 0.00006047116070 \\
-3 & 0.0001040735229 & 0.00009787945935 & 0.00006077519798 \\
\hline
\end{array}
\]
5.3 Error values of the 8-point method for the second solution

Table 17

<table>
<thead>
<tr>
<th>μ</th>
<th>ν = 10^{-5}</th>
<th>ν = 10^{-4}</th>
<th>ν = 75 \times 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.000007460217203</td>
<td>0.00004310232029</td>
<td>0.0003494254334</td>
</tr>
<tr>
<td>3</td>
<td>0.000007579286198</td>
<td>0.00004379297289</td>
<td>0.0003538533044</td>
</tr>
<tr>
<td>-2</td>
<td>0.000004224021082</td>
<td>0.00001344785629</td>
<td>0.0001107447140</td>
</tr>
<tr>
<td>-3</td>
<td>0.00000534356399</td>
<td>0.00002173540856</td>
<td>0.0001865530256</td>
</tr>
</tbody>
</table>

Table 18

<table>
<thead>
<tr>
<th>μ</th>
<th>ν = 10^{-5}</th>
<th>ν = 10^{-4}</th>
<th>ν = 75 \times 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.000003873177497</td>
<td>0.00002527495600</td>
<td>0.002011419919</td>
</tr>
<tr>
<td>3</td>
<td>0.00003988029855</td>
<td>0.0002553613352</td>
<td>0.002029365543</td>
</tr>
<tr>
<td>-2</td>
<td>0.00002328543947</td>
<td>0.0001060633180</td>
<td>0.0008836148448</td>
</tr>
<tr>
<td>-3</td>
<td>0.00002894850318</td>
<td>0.0001588996524</td>
<td>0.001294451285</td>
</tr>
</tbody>
</table>

5.4 Error values of the 12-point method for the second solution

Table 19

<table>
<thead>
<tr>
<th>μ</th>
<th>ν = 10^{-5}</th>
<th>ν = 10^{-4}</th>
<th>ν = 75 \times 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.000003972635067</td>
<td>0.00001817379673</td>
<td>0.0001051254334</td>
</tr>
<tr>
<td>3</td>
<td>0.00002645273873</td>
<td>0.00001784337140</td>
<td>0.0001052533044</td>
</tr>
<tr>
<td>-2</td>
<td>0.000002771518846</td>
<td>0.000001778433723</td>
<td>0.0001063728472</td>
</tr>
<tr>
<td>-3</td>
<td>0.000002604627907</td>
<td>0.00001875348618</td>
<td>0.0001057530256</td>
</tr>
</tbody>
</table>

Table 20

<table>
<thead>
<tr>
<th>μ</th>
<th>ν = 10^{-5}</th>
<th>ν = 10^{-4}</th>
<th>ν = 75 \times 10^{-5}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.00001540343599</td>
<td>0.00000983537861</td>
<td>0.0006037376406</td>
</tr>
<tr>
<td>3</td>
<td>0.000009686174560</td>
<td>0.00009783111057</td>
<td>0.0006055914273</td>
</tr>
<tr>
<td>-2</td>
<td>0.00001068116463</td>
<td>0.00009674956640</td>
<td>0.000609170592</td>
</tr>
<tr>
<td>-3</td>
<td>0.000009698413538</td>
<td>0.00009786692026</td>
<td>0.0006072799733</td>
</tr>
</tbody>
</table>

By looking at the tables, we conclude that for equal values of μ, as ν increases, the error values (∥E∥₂ or ∥E∥∞) increase as well. In the forthcoming article, a problem regarding the traffic is discussed. Then, this issue is modeled by using Burgers’ equation. Also, Burgers’ equation is formulated by using the above mentioned multisymplectic and Runge-kutta 4 methods. Finally, the results are evaluated.
Fig. 1. Error diagram of 8-point multisymplectic box method for table 2 with $\mu = -3$

Fig. 2. Error diagram of 12-point multisymplectic box method for table 5 with $\mu = -3$
Fig. 3. Error diagram of 8-point multisymplectic box method for table 9 with $\mu = 2$

Fig. 4. Error diagram of 12-point multisymplectic box method for table 11 with $\mu = -3$
References


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