Continuous Monotonic Decomposition of
Some Special Class of Graphs

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Abstract. The concept of continuous monotonic decomposition (CMD) was introduced by Paulraj Joseph and Gnanadhas[2]. A graph $G$ of size $q=\binom{n+1}{2}$ is said to have a CMD if $G$ can be decomposed into $n$-subgraphs $G_1, G_2, \ldots G_n$ such that each $G_i$ is connected and $|E(G_i)| = i$ for $1 \leq i \leq n-1$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The tensor product $G = G_1 \wedge G_2$ is defined as a graph with vertex set $V_1 \times V_2$. Edge set is defined as follows: If $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ are two vertices of $G$ with $u_i \in V_1$ and $v_i \in V_2$, ($i = 1, 2$) then $w_1w_2 \in E(G)$ if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$. In this paper, we investigate CMD for some special class of graphs, namely $P_n \wedge K_2$, $C_n \wedge K_2$, $K_{1,n} \wedge K_2$ and $W_n \wedge K_2$.

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1. Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. A wheel on $n$ vertices is denoted by $W_n$. A path of length $n$ is denoted
Definition 1.2. A graph $G$ of size $q=\binom{n+1}{2}$ is said to have a continuous monotonic decomposition (CMD) if $G$ can be decomposed into $n$-subgraphs $G_1, G_2, \ldots, G_n$ such that each $G_i$ is connected and $|E(G_i)| = i$ for each $i = 1, 2, 3, \ldots, n$.

Definition 1.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The tensor product $G = G_1 \otimes G_2$ is defined as a graph with vertex set $V_1 \times V_2$. Edge set is defined as follows; If $v_1 = (u_1, v_1)$ and $v_2 = (u_2, v_2)$ are two vertices of $G$ with $u_i \in V_1$ and $v_i \in V_2$ ($i = 1, 2$) then $v_1v_2 \in E(G)$ if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$.

We use the following theorem.

Theorem 1.4. [2] Let $G$ be a connected simple graph of order $p$ and size $q$. Then $G$ admits a CMD if and only if $q = \binom{n+1}{2}$.

CMD of some class of Graphs

We now investigate the cases under which some special class of graphs have CMD.

2. CMD of $P_n \otimes K_2$

In this section, we prove $P_n \otimes K_2$ admits CMD under certain conditions. We prove two lemmas to exhibit a CMD of $P_n \otimes K_2$ in theorem 2.3.

Lemma 2.1. Let $m \equiv 0 \pmod{4}$. The set $\{1, 2, \ldots, m\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1$. Here $\frac{m(m+1)}{2} = 2n - 2$.

Proof. Let $m = 4k$, $k \geq 1, k \in \mathbb{Z}$. Proof is by induction on $k$. When $k = 1, m = 4$. Now $n = \frac{1}{2}[2 + \frac{m(m+1)}{2}] = 6$. If $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$ then $\sum_{x \in S_1} x = 1 + 4 = 5 = 6 - 1$ and $\sum_{y \in S_2} y = 2 + 3 = 5 = 6 - 1$. Hence the result is true if $k = 1$. Assume that the result is true for $k - 1$. Hence the set $\{1, 2, \ldots, 4(k-1)\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1 = (4k-3)(k-1)$. To prove the result is true for $k$. The set $\{1, 2, \ldots, 4k\}$ can be partitioned into two sets $S_1'$ and $S_2'$ where
\[ S'_1 = S_1 \cup \{4k - 3, 4k\} \text{ and } S'_2 = S_2 \cup \{4k - 2, 4k - 1\}. \]

Clearly
\[
\sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4k - 3 + 4k = (4k - 3)(k - 1) + 4k - 3 + 4k = 4k^2 + k = k(4k + 1)
\]
\[ = n - 1 \]

Similarly
\[ \sum_{y \in S'_2} y = n - 1 \]

Hence the lemma is proved for all \(k\). \(\square\)

**Lemma 2.2.** Let \(m + 1 \equiv 0 \pmod{4}\). The set \(\{1, 2, \ldots, m\}\) can be partitioned into two sets \(S_1\) and \(S_2\) such that \(\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1\). Here \(\frac{m(m+1)}{2} = 2n - 2\).

**Proof.** Let \(m + 1 = 4k\), \(k \geq 1, k \in \mathbb{Z}\) and so \(m = 4k - 1\). Proof is by induction on \(k\). When \(k = 1, m = 3\). Now \(n = 4\). If \(S_1 = \{1, 2\}\) and \(S_2 = \{3\}\) then \(\sum_{x \in S_1} x = 3 = 4 - 1\) and \(\sum_{y \in S_2} y = 3 = 4 - 1\). Hence the result is true if \(k = 1\). Assume that the result is true for \(k - 1\). Hence the set \(\{1, 2, \ldots, (4(k - 1) - 1)\}\) can be partitioned into two sets \(S_1\) and \(S_2\) such that \(\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1 = (k - 1)(4k - 5)\). To prove the result is true for \(k\). The set \(\{1, 2, \ldots, 4k - 1\}\) can be partitioned into two sets \(S'_1\) and \(S'_2\) where \(S'_1 = S_1 \cup \{4k - 4, 4k - 1\}\) and \(S'_2 = S_2 \cup \{4k - 3, 4k - 2\}\).

Now
\[
\sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4k - 4 + 4k - 1 = (4k - 5)(k - 1) + 4k - 4 + 4k - 1 = 4k^2 - k = k(4k - 1) = n - 1
\]
\[
\sum_{y \in S'_2} y = \sum_{y \in S_2} y + 4k - 3 + 4k - 2 = (4k - 5)(k - 1) + 8k - 5 = 4k^2 - k = k(4k - 1) = n - 1
\]
Hence by induction, the lemma is true for all \( k \).

**Theorem 2.3.** For any integer \( n \), \( P_n \cap K_2 \) has a CMD \( \{ H_1, H_2, \ldots, H_m \} \) if and only if there exists an integer \( m \) satisfying the following properties:

(i) \( m = 4k \) or \( 4k - 1 \) \( (k \geq 1, k \in \mathbb{Z}) \)

(ii) \( \frac{m(m+1)}{2} = 2n - 2 \).

**Proof.** Let \( G = P_n \cap K_2 \). By definition, \( |E(G)| = 2n - 2 \). Assume \( P_n \cap K_2 \) has a CMD \( \{ H_1, H_2, \ldots, H_m \} \). Now by theorem 1.4, \( |E(G)| = \binom{m+1}{2} \). Hence \( 2n - 2 = \binom{m+1}{2} \), i.e., \( 2n - 2 = \frac{m(m+1)}{2} \). Since \( P_n \cap K_2 \) has a CMD, \( 2n - 2 = 1 + 2 + \ldots + m \)

\[ \Rightarrow 2(n - 1) = \frac{m(m + 1)}{2} \]

\[ \Rightarrow m(m + 1) = 4(n - 1) \]

Hence \( m(m + 1) \equiv 0 \pmod{4} \)

\[ \Rightarrow m(m + 1) = 4k \ (k \geq 1, k \in \mathbb{Z}) \]

Now either \( m = 4k \) or \( m + 1 = 4k \). Thus \( m = 4k \) or \( m = 4k - 1 \), where \( k \geq 1, k \in \mathbb{Z} \).

Conversely assume \( m(m + 1) \equiv 0 \pmod{4} \). Let \( G = P_n \cap K_2 \). Let \( P_n = \{ u_1, u_2, \ldots, u_n \} \) and \( K_2 = \{ v_1, v_2 \} \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n, 1 \leq j \leq 2 \). Now \( V(G) = \{ w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2 \} \) and \( |E(G)| = 2n - 2 \).

**Case (i):** Suppose \( n \) is even.

Define \( T_1 = \{ w_{i1}, w_{i(i+1)2} : 1 \leq i \leq n, i \text{ is odd} \} \) and \( T_2 = \{ w_{i2}, w_{i(i+1)1} : 1 \leq i \leq n, i \text{ is even} \} \).

Also, \( |T_1| = n - 1 \) and \( |T_2| = n - 1 \). Thus \( |T_1| + |T_2| = n - 1 \). Also, \( |T_1| + |T_2| = 1 + 2 + \ldots + m = \binom{m+1}{2} \). By lemmas 2.1 and 2.2, \( \{ 1, 2, 3 \ldots, m \} = S_1 \cup S_2 \) where \( \sum_{x \in S_1} x = n - 1 \) and \( \sum_{y \in S_2} y = n - 1 \). Decompose \( T_1, T_2 \) into trees \( \{ H_i \} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i, T_2 = \bigcup_{i \in S_2} H_i \) and \( |E(H_i)| = i, 1 \leq i \leq m \). Clearly \( \{ H_1, H_2, \ldots, H_m \} \) forms a CMD of \( P_n \cap K_2 \).

**Case (ii):** Suppose \( n \) is odd.

Define \( T_1 = \{ w_{i1}, w_{i(i+1)2} : 1 \leq i \leq n - 1, i \text{ is odd} \} \) and \( T_2 = \{ w_{i2}, w_{i(i+1)1} : 1 \leq i \leq n, i \text{ is even} \} \).

Also, \( |T_1| = n - 1 \) and \( |T_2| = n - 1 \). Thus \( |T_1| + |T_2| = n - 1 \). Also, \( |T_1| + |T_2| = 1 + 2 + \ldots + m = \binom{m+1}{2} \). By lemmas 2.1 and 2.2, \( \{ 1, 2, 3 \ldots, m \} = S_1 \cup S_2 \) where \( \sum_{x \in S_1} x = n - 1 \) and \( \sum_{y \in S_2} y = n - 1 \). Decompose \( T_1, T_2 \) into trees \( \{ H_i \} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i \) and \( T_2 = \bigcup_{i \in S_2} H_i \). Clearly \( \{ H_1, H_2, \ldots, H_m \} \) forms a CMD of \( P_n \cap K_2 \).

**Illustration 2.4.** As an illustration let us decompose \( P_6 \cap K_2 \). Let \( V(P_6) = \{ u_1, u_2, \ldots, u_6 \} \). Let \( V(K_2) = \{ v_1, v_2 \} \). \( P_6 \cap K_2 \) is given in figure 1.
Continuous monotonic decomposition

$P_6 \land K_2$:

\[
\begin{array}{cccccc}
(u_1, v_1) & (u_2, v_1) & (u_3, v_1) & (u_4, v_1) & (u_5, v_1) & (u_6, v_1) \\
(u_1, v_2) & (u_2, v_2) & (u_3, v_2) & (u_4, v_2) & (u_5, v_2) & (u_6, v_2)
\end{array}
\]

*Figure 1*

Here $|E(G)| = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, v_2))$ where $1 \leq i, j \leq 6$. $T_1 = \{e_{12}, e_{32}, e_{34}, e_{54}, e_{56}\}$ and $T_2 = \{e_{21}, e_{23}, e_{43}, e_{45}, e_{65}\}$. $|T_1| = 5 = |T_2|$. Hence $|T_1| + |T_2| = 10 = 1 + 2 + 3 + 4 = \binom{6}{2}$. $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. $T_1$ is decomposed as, $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$ and $H_4 = \{e_{32}, e_{34}, e_{54}, e_{56}\}$. $T_2$ is decomposed as, $T_2 = H_2 \cup H_3$ where $H_2 = \{e_{21}, e_{23}\}$ and $H_3 = \{e_{43}, e_{45}, e_{65}\}$. $\{H_1, H_2, H_3, H_4\}$ forms a CMD of $P_6 \land K_2$.

3. **CMD of $C_n \land K_2$ and $K_{1,n} \land K_2$**

**Lemma 3.1.** Let $m \equiv 0 \pmod{4}$. The set $\{1, 2, 3, \ldots, m\}$ can be partitioned into two sets $S_1, S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n$. Here \(\frac{m(m+1)}{2} = 2n\).

*Proof.* Let $m = 4k$, $k \geq 1, k \in \mathbb{Z}$. Proof is by induction on $k$. When $k = 1, m = 4$ and $n = 5$. Let $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. Now $\sum_{x \in S_1} x = 1 + 4 = 5$ and $\sum_{y \in S_2} y = 2 + 3 = 5 = n$ so that the result is true if $k = 1$. Assume that the result is true for $k - 1$. Hence the set $\{1, 2, 3, \ldots, 4(k-1)\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n = (k-1)(4k-3)$. To prove the result is true for $k$. The set $\{1, 2, 3, \ldots, 4k\}$ can be partitioned into
two sets $S'_1$ and $S'_2$ where $S'_1 = S_1 \cup \{4k-3, 4k\}$ and $S'_2 = S_2 \cup \{4k-2, 4k-1\}$.

Now \[ \sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4k - 3 + 4k \]
\[ = (k - 1)(4k - 3) + 4k - 3 + 4k \]
\[ = 4k^2 + k \]
\[ = n. \]

Also \[ \sum_{y \in S'_2} y = \sum_{y \in S_2} y + 4k - 2 + 4k - 1 \]
\[ = (k - 1)(4k - 3) + 4k - 2 + 4k - 1 \]
\[ = 4k^2 + k \]
\[ = n. \]

Hence by induction the lemma is true for all $k$. \qed

**Lemma 3.2.** Let $m + 1 \equiv 0 \pmod{4}$. The set $\{1, 2, 3, \ldots, m\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum x = \sum y = n$. Here $\frac{m(m+1)}{2} = 2n$.

**Proof.** Let $m + 1 = 4k$, $k \geq 1, k \in \mathbb{Z}$ so that $m = 4k - 1$. Proof is by induction on $k$. When $k = 1, m = 3$ and $n = 3$. Let $S_1 = \{1, 2\}$ and $S_2 = \{3\}$. Now $\sum x = 1 + 2 = 3 = n$ and $\sum y = 3 = n$. Hence the result is true if $k = 1$. Assume that the result is true for $k - 1$. Hence the set $\{1, 2, 3, \ldots, 4(k - 1) - 1\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum x = \sum y = n = (k - 1)(4k - 5)$. To prove the result is true for $k$.

The set $\{1, 2, 3, \ldots, 4k - 1\}$ can be partitioned into two sets $S'_1$ and $S'_2$ where $S'_1 = S_1 \cup \{4k - 4, 4k - 1\}$ and $S'_2 = S_2 \cup \{4k - 3, 4k - 2\}$.

Now \[ \sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4k - 4 + 4k - 1 \]
\[ = (k - 1)(4k - 5) + 4k - 4 + 4k - 1 \]
\[ = 4k^2 - k \]
\[ = n. \]

Also \[ \sum_{y \in S'_2} y = \sum_{y \in S_2} y + 4k - 3 + 4k - 2 \]
\[ = (k - 1)(4k - 5) + 8k - 5 \]
\[ = 4k^2 - k \]
\[ = n. \]

Hence by induction the lemma is true for all $k$. \qed
Theorem 3.3. For any integer \( n \), \( C_n \cap K_2 \) has a CMD \( \{H_1, H_2, \ldots, H_m\} \) if and only if there exists an integer \( m \) satisfying the following properties:
(i) \( m = 4k \) or \( 4k - 1 \) \( (k \geq 1, k \in \mathbb{Z}) \)
(ii) \( \frac{m(m+1)}{2} = 2n \).

Proof. Let \( G = C_n \cap K_2 \). By definition, \( |E(G)| = 2n \). Assume \( C_n \cap K_2 \) has a CMD. By theorem 1.4, \( |E(G)| = \binom{m+1}{2} \). Hence \( 2n = \binom{m+1}{2} = \frac{m(m+1)}{2} \).
Since \( C_n \cap K_2 \) has a CMD,
\[
2n = 1 + 2 + 3 + \ldots + m
\]
\[
\Rightarrow 2n = \frac{m(m+1)}{2}
\]
\[
\Rightarrow m(m+1) = 4n
\]
Hence \( m(m+1) \equiv 0 \pmod{4} \)
\[
\Rightarrow m(m+1) = 4k
\]
\[
\Rightarrow m = 4k \) or \( m + 1 = 4k
\]
\[
\Rightarrow m = 4k \) or \( m = 4k - 1 \), where \( k \geq 1, k \in \mathbb{Z} \)

Conversely, assume \( m(m+1) \equiv 0 \pmod{4} \). Let \( G = C_n \cap K_2 \). Let \( C_n = (u_1, u_2, \ldots, u_n, u_1) \) and \( K_2 = (v_1, v_2) \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n, 1 \leq j \leq 2 \). Now \( V(G) = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\} \) and \( |E(G)| = 2n \).

Case(i): Suppose \( n \) is even.
Define \( T_1 = \{(w_{i1}, w_{i(i+1)2}) : 1 \leq i \leq n, i \text{ - odd}\} \cup \{(w_{i2}, w_{i(i+1)1}) : 1 \leq i \leq n - 1, i \text{ - even}\} \cup \{(w_{i2}, w_{11}) : i = n\} \) and \( T_2 = \{(w_{i2}, w_{i(i+1)1}) : 1 \leq i \leq n, i \text{ - odd}\} \cup \{(w_{i1}, w_{i(i+1)2}) : 1 \leq i \leq n - 1, i \text{ - even}\} \cup \{(w_{11}, w_{12}) : i = n\} \).
Here \( |T_1| = n \) and \( |T_2| = n \). Also, \( |T_1| + |T_2| = 1 + 2 + 3 + \ldots + m = \binom{m+1}{2} \) By lemmas 3.1 and 3.2, \( \{1, 2, 3, \ldots, m\} = S_1 \cup S_2 \) where \( x \in S_1 \) and \( y \in S_2 \).

Decompose \( T_1 \) and \( T_2 \) into trees \( \{H_i\} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i \), \( T_2 = \bigcup_{i \in S_2} H_i \) and \( |E(H_i)| = i, 1 \leq i \leq m \). Clearly \( \{H_1, H_2, \ldots, H_m\} \) forms a CMD of \( C_n \cap K_2 \).

Case(ii): Suppose \( n \) is odd.
Define \( T_1 = \{(w_{i1}, w_{i(i+1)2}) : 1 \leq i \leq n - 1, i \text{ - odd}\} \cup \{(w_{i2}, w_{i(i+1)1}) : 1 \leq i \leq n, i \text{ - even}\} \cup \{(w_{i1}, w_{12}) : i = n\} \) and \( T_2 = \{(w_{i2}, w_{i(i+1)1}) : 1 \leq i \leq n - 1, i \text{ - odd}\} \cup \{(w_{i1}, w_{i(i+1)2}) : 1 \leq i \leq n, i \text{ - even}\} \cup \{(w_{11}, w_{12}) : i = n\} \).
Here \( |T_1| = n \) and \( |T_2| = n \). Also, \( |T_1| + |T_2| = 1 + 2 + 3 + \ldots + m = \binom{m+1}{2} \).
By lemma 3.1 and 3.2, \( \{1, 2, 3, \ldots, m\} = S_1 \cup S_2 \) where \( x \in S_1 \) and \( y \in S_2 \).

Decompose \( T_1 \) and \( T_2 \) into trees \( \{H_i\} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i \), \( T_2 = \bigcup_{i \in S_2} H_i \) and \( |E(H_i)| = i, 1 \leq i \leq m \). Clearly \( \{H_1, H_2, \ldots, H_m\} \) forms a CMD of \( C_n \cap K_2 \). □
Illustration 3.4. As an illustration, let us decompose $C_5 \land K_2$. Let $V(C_5) = \{u_1, u_2, \ldots, u_5\}$ and $V(K_2) = \{v_1, v_2\}$. $C_5 \land K_2$ is given in Figure 2.

$C_5 \land K_2$:

![Graph of $C_5 \land K_2$](image)

Here $|E(G)| = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, v_2))$, where $1 \leq i, j \leq 5$.

$T_1 = \{e_{12}, e_{32}, e_{34}, e_{54}, e_{51}\}$, $T_2 = \{e_{21}, e_{23}, e_{43}, e_{45}, e_{15}\}$. $|T_1| = |T_2| = 5$. Hence $T_1 | T_2 | T_2 = 10 = 1 \cdot 2 + 3 + 4 = \binom{5}{2}$. $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. $T_1$ is decomposed as $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$ and $H_2 = \{e_{32}, e_{34}, e_{54}, e_{51}\}$. $T_2$ is decomposed as $T_2 = H_2 \cup H_3$ where $H_2 = \{e_{21}, e_{23}\}$ and $H_3 = \{e_{43}, e_{45}, e_{15}\}$, $\{H_1, H_2, H_3, H_4\}$ forms a CMD of $C_5 \land K_2$.

Theorem 3.5. For any integer $n$, $K_{1,n} \land K_2$ has a CMD if and only if there exists an integer $m$ satisfying the following properties:

(i) $m = 4k$ or $4k - 1 (k \geq 1, k \in \mathbb{Z})$

(ii) $\frac{m(m+1)}{2} = 2n$

Proof. Let $G = K_{1,n} \land K_2$. By definition, $|E(G)| = 2n$. Assume $K_{1,n} \land K_2$ has a CMD. By theorem 1.4, $|E(G)| = \binom{m+1}{2}$. Hence $2n = \binom{m+1}{2} = \frac{m(m+1)}{2}$. Since $K_{1,n} \land K_2$ has a CMD,

$$\Rightarrow 2n = 1 + 2 + 3 + \ldots + m$$
$$\Rightarrow 2n = \frac{m(m+1)}{2}$$
$$\Rightarrow m(m+1) = 4n$$

Hence $m(m+1) \equiv 0 (mod \ 4)$

$$\Rightarrow m(m+1) = 4k$$
$$\Rightarrow m = 4k \ or \ m + 1 = 4k$$
$$\Rightarrow m = 4k \ or \ m = 4k - 1, \ where \ k \geq 1, k \in \mathbb{Z}$$

Conversely assume $m(m+1) \equiv 0 (mod \ 4)$. Let $G = K_{1,n} \land K_2$. Let $K_{1,n}$ be the $n$ -star with $u_1$ as the center and pendents denoted by $u_2, u_3, \ldots, u_{n+1}$. Let
$K_2 = (v_1, v_2)$. Define $w_{ij} = (u_i, v_j)$ where $1 \leq i \leq n+1$, $1 \leq j \leq 2$. Now $V(G) = \{w_{ij} : 1 \leq i \leq n+1, 1 \leq j \leq 2\}$ and $|E(G)| = 2n$. For every integer $n \in \mathbb{Z}$, define $T_1 = \{w_{11}, w_{(i+1)2} : 1 \leq i \leq n\}$ and $T_2 = \{w_{12}, w_{(i+1)1} : 1 \leq i \leq n\}$. Here $|T_1| = n$ and $|T_2| = n$. Also, $|T_1| + |T_2| = 1 + 2 + \ldots + m = \left(\frac{m+1}{2}\right)$. By lemmas 3.1 and 3.2, $\{1, 2, \ldots, m\} = S_1 \cup S_2$, where $\sum x = n$ and $\sum y = n$.

Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \bigcup_{i \in S_1} H_i$, $T_2 = \bigcup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq m$. Clearly $\{H_1, H_2, \ldots, H_m\}$ forms a CMD of $K_{1,n} \land K_2$.

**Illustration 3.6.** As an illustration let us decompose $K_{1,5} \land K_2$. Let $V(K_{1,5}) = \{u_1, u_2, \ldots, u_6\}$ where $u_1$ is the center of $K_{1,5}$. Let $K_2 = (v_1, v_2)$. $K_{1,5} \land K_2$ is given in figure 3.

$K_{1,5}:$

```
        u1
       /|
   u2  u3  u4  u5  u6
```

$K_2: v1 -- v2$

$K_{1,5} \land K_2:$

```
(u1, v1)  (u1, v2)  (u2, v1)  (u2, v2)  (u3, v1)  (u3, v2)  (u4, v1)  (u4, v2)  (u5, v1)  (u5, v2)  (u6, v1)  (u6, v2)
```

*Figure 3*

Here $|E(G)| = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, v_2))$, where $1 \leq i, j \leq 6$. $T_1 = \{e_{12}, e_{13}, e_{14}, e_{15}, e_{16}\}$, $T_2 = \{e_{21}, e_{31}, e_{41}, e_{51}, e_{61}\}$. $|T_1| = |T_2| = 5$. Hence $|T_1| = |T_2| = 10 = 1 + 2 + 3 + 4 = \binom{5}{2}$. $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. $T_1$ is decomposed as, $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$ and $H_4 = \{e_{13}, e_{14}, e_{15}, e_{16}\}$. $T_2$ is decomposed as, $T_2 = H_2 \cup H_3$ where $H_2 = \{e_{21}, e_{31}\}$ and $H_3 = \{e_{41}, e_{51}, e_{61}\}$. $\{H_1, H_2, H_3, H_4\}$ forms a CMD of $K_{1,5} \land K_2$.

4. CMD of $W_{n+1} \land K_2$

**Lemma 4.1.** Let $m \equiv 0 \pmod{8}$. The set $\{1, 2, \ldots, m\}$ can be partitioned into two sets $S_1, S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n$. Here $\frac{m(m+1)}{2} = 4n$. 
Proof. Let \( m = 8k, \ k \geq 1, \ k \in \mathbb{Z}. \)

Proof is by induction on \( k. \) When \( k = 1, m = 8 \) and \( n = 9. \) Let \( S_1 = \{1, 4, 5, 8\} \) and \( S_2 = \{2, 3, 6, 7\}. \) Now \( \sum_{x \in S_1} x = 1 + 4 + 5 + 8 = 18 = 2n \) and \( \sum_{y \in S_2} y = 2 + 3 + 6 + 7 = 2n. \) Hence the result is true if \( k = 1. \) Assume the result is true for \( k - 1. \) Hence the set \( \{1, 2, \ldots, 8(k - 1)\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n = (2k - 2)(8k - 7). \)

To prove the result is true for \( k. \) The set \( \{1, 2, 3, \ldots, 8k\} \) can be partitioned into two sets \( S_1' \) and \( S_2' \) where \( S_1' = S_1 \cup \{8k, 8k - 2, 8k - 5, 8k - 7\} \) and \( S_2' = S_2 \cup \{8k - 1, 8k - 3, 8k - 4, 8k - 6\}. \)

Now \( \sum_{x \in S_1'} x = \sum_{x \in S_1} x + 8k + 8k - 2 + 8k - 5 + 8k - 7 \)
\[= (2k - 2)(8k - 7) + 32k - 14 \]
\[= 16k^2 + 2k \]
\[= 2n. \]

Also \( \sum_{y \in S_2'} y = \sum_{y \in S_2} y + 8k - 1 + 8k - 3 + 8k - 4 + 8k - 6 \)
\[= (2k - 2)(8k - 7) + 32k - 14 \]
\[= 16k^2 + 2k \]
\[= 2n. \]

Hence the lemma is proved for all \( k. \)

\[\square\]

**Lemma 4.2.** Let \( m + 1 \equiv 0 \pmod{8}. \) The set \( \{1, 2, \ldots, m\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n. \) Here \( \frac{m(m + 1)}{2} = 4n. \)

Proof. Let \( m + 1 = 8k, \ k \geq 1, \ k \in \mathbb{Z} \) so that \( m = 8k - 1. \)

Proof is by induction on \( k. \) When \( k = 1, m = 7 \) and \( n = 7. \) Let \( S_1 = \{1, 2, 4, 7\} \) and \( S_2 = \{3, 5, 6\}. \) Now \( \sum_{x \in S_1} x = 1 + 2 + 4 + 7 = 14 = 2n \) and \( \sum_{y \in S_2} y = 3 + 5 + 6 = 14 = 2n. \) Hence the result is true if \( k = 1. \) Assume the result is true for \( k - 1. \) Hence the set \( \{1, 2, \ldots, (8(k - 1) - 1)\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n = (2k - 2)(8k - 9). \)

To prove the result is true for \( k. \) The set \( \{1, 2, 3, \ldots, 8k - 1\} \) can be partitioned into sets \( S_1' \) and \( S_2' \) where \( S_1' =\)
Define \( w \) odd

Hence the lemma is proved for all \( k \).

**Theorem 4.3.** For any integer \( n \), \( W_{n+1} \wedge K_2 \) has a CMD \( \{ H_1, H_2, \ldots, H_m \} \) if and only if there exists an integer \( m \) satisfying the following properties:

(i) \( m = 8k \) or \( 8k - 1 \) \((k \geq 1, \ k \in \mathbb{Z})\)

(ii) \( \frac{m(m+1)}{2} = 4n \)

**Proof.** Let \( G = W_{n+1} \wedge K_2 \). By definition, \( |E(G)| = 4n \). Assume \( W_{n+1} \wedge K_2 \) has a CMD. By Theorem 1.4, \( |E(G)| = \binom{m+1}{2} \). Hence \( 4n = \binom{m+1}{2} = \frac{m(m+1)}{2} \).

Since \( W_{n+1} \wedge K_2 \) has a CMD,

\[
4n = 1 + 2 + \ldots + m
\]

\[
\Rightarrow 4n = \frac{m(m+1)}{2}
\]

\[
\Rightarrow m(m+1) = 8n.
\]

Hence \( m(m+1) \equiv 0 (\text{mod } 8) \)

\[
\Rightarrow m(m+1) = 8k
\]

\[
\Rightarrow m = 8k \text{ or } m + 1 = 8k
\]

\[
\Rightarrow m = 8k \text{ or } m = 8k - 1 \text{ where } k \geq 1, \ k \in \mathbb{Z}
\]

Conversely, assume \( m(m+1) \equiv 0 (\text{mod } 8) \). Let \( G = W_{n+1} \wedge K_2 \). Here \( W_{n+1} \) is a wheel on \( n + 1 \) vertices with \( u_1, u_2, \ldots, u_n \) representing the vertices on the cycle and \( u_{n+1} \) representing the vertex of degree \( n \). Let \( K_2 = (v_1, v_2) \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n + 1, \ 1 \leq j \leq 2 \). Now \( V(G) = \{ w_{ij} : 1 \leq i \leq n + 1, \ 1 \leq j \leq 2 \} \) and \( |E(G)| = 4n \). For every integer \( n \in \mathbb{Z} \), define \( T_1 = \{ w_{i1}, w_{i(n+1)/2} : 1 \leq i \leq n, i - \text{odd} \} \cup \{ (w_{i1}, w_{(n+1)/2}) : 1 \leq i \leq n, i - \text{odd} \} \cup \{ (w_{i+1}, w_{1}) : 1 \leq i < n, i - \text{even} \} \cup \{ (w_{i1}, w_{1}) : 1 \leq i \leq n, i - \text{odd} \} \) and \( T_2 = \{ w_{i1}, w_{(i+1)/2} : 1 \leq i < n, i - \text{even} \} \cup \{ (w_{i1}, w_{(n+1)/2}) : 1 \leq i \leq n, i - \text{odd} \} \).
\(n, i - \text{even} \} \cup \{w_{i+1}, w_i : 1 \leq i < n, i - \text{even} \} \cup \{w_{(n+1)/2}, w_{1/2} : 1 \leq i \leq n, i - \text{even} \} \cup \{w_1, w_{i/2} : i = n \} \cup \{w_{11}, w_{i/2} : i = n \}. \) Here |\(T_1| = 2n\) and |\(T_2| = 2n.\) Also, |\(T_1| + |T_2| = 1 + 2 + \ldots + m = \binom{m+1}{2}.\) By lemmas 4.1 and 4.2, \(\{1, 2, 3 \ldots m\} = S_1 \cup S_2\) where \(\sum x = 2n\) and \(\sum y = 2n.\)

Decompose \(T_1\) and \(T_2\) into trees \(\{H_i\}\) as follows: \(T_1 = \bigcup_{i \in S_1} H_i, T_2 = \bigcup_{i \in S_2} H_i\) and |\(E(H_i)| = i, 1 \leq i \leq m.\) Clearly \(\{H_1, H_2, \ldots, H_m\}\) forms a CMD of \(W_{n+1} \land K_2.\)

**Illustration 4.4.** As an illustration let us decompose \(W_8 \land K_2.\) Let \(V(W_8) = \{u_1, u_2, \ldots, u_8\}\) where \(u_1, u_2, \ldots, u_7\) are the vertices on the cycle and \(u_8\) represents the vertex of degree 7. Let \(K_2 = \{v_1, v_2\}.\) \(W_8 \land K_2\) is given in Figure 4.

![Figure 4](image)

Here |\(E(G)| = 28\) and \(m = 7.\) Let \(e_{ij} = ((u_i, v_1), (u_j, v_2)). T_1 = \{e_{12}, e_{34}, e_{56}, e_{18}, e_{38}, e_{58}, e_{78}, e_{21}, e_{43}, e_{65}, e_{81}, e_{83}, e_{85}, e_{87}\}. T_2 = \{e_{17}, e_{23}, e_{28}, e_{32}, e_{45}, e_{48}, e_{54}, e_{67}, e_{68}, e_{71}, e_{76}, e_{82}, e_{84}, e_{86}\}. |T_1| = |T_2| = 14. \) Hence |\(T_1| = |T_2| = 28 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = \binom{7}{2}.\) Here \(S_1 = \{1, 2, 4, 7\}\) and \(S_2 = \{3, 5, 6\}.\) \(T_1\) is decomposed as \(T_1 = H_1 \cup H_2 \cup H_4 \cup H_7\) where \(H_1 = \{e_{56}\}, H_2 = \{e_{12}, e_{18}\}, H_4 = \{e_{34}, e_{38}, e_{58}, e_{78}\}\) and \(H_7 = \{e_{21}, e_{43}, e_{65}, e_{81}, e_{83}, e_{85}, e_{87}\}. T_2\) is decomposed as \(T_2 = H_3 \cup H_5 \cup H_6\) where \(H_3 = \{e_{67}, e_{71}, e_{17}\}, H_5 = \{e_{23}, e_{28}, e_{45}, e_{48}, e_{68}\}\) and \(H_6 = \{e_{32}, e_{54}, e_{76}, e_{82}, e_{84}, e_{86}\}.\) Clearly \(\{H_1, H_2, \ldots, H_7\}\) forms a CMD of \(W_n \land K_2.\)

**References**


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