On the Convergence of Ishikawa Type Iteration with Errors to a Common Fixed Point of Two Mappings in Convex Metric Spaces

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Abstract

In this paper, we study the Ishikawa type iteration process with errors, which converges to a common fixed point of a pair of mappings in complete generalized convex metric spaces. Furthermore, we obtain the corresponding results in Banach spaces. Our result improve and generalize a recent result of Khan[S.H. Khan, Common fixed points of two quasi-contractive operators in normed spaces by iteration, Int.Journal of Math.Analysis,3(3)(2009) 145-151] and others.

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1 Introduction and Preliminaries

In recent years, many authors have studied the iterative approximation of fixed points of a mapping $T$ by using some iteration processes, such as Krasnoselskij\cite{1}, Mann and Ishikawa iteration method\cite{2, 3, 4}, Mann and Ishikawa iteration process with errors in the sense of Liu\cite{5} or Xu\cite{6}, two-step iteration \cite{7, 8} and implicit iteration method\cite{9, 10, 11}etc.,

Let $(X, d)$ be a metric space, $C$ be a nonempty subset of $X$. We use $F(T)$ to denote the set of fixed points of $T$, i.e., $F(T) = \{x \in X : Tx = x\}$. 


A mapping \( T : C \to C \) is called \( a \)-contraction, if
\[
d(Tx, Ty) \leq ad(x, y), \quad \text{for all } x, y \in C,
\]
where \( 0 < a < 1 \). The map \( T \) is called Kannan[12] mapping if there exists \( b \in (0, 1/2) \) such that
\[
d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in C.
\]
The map \( T \) is called Chatterjea[13] mapping if there exists \( c \in (0, 1/2) \) such that
\[
d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \quad \text{for all } x, y \in C.
\]
Zamfirescu[14] combined the above definitions and proved the following theorem.

**Theorem 1.1** ([14]) Let \((X, d)\) be a complete metric space and \( T : X \to X \) a mapping for which there exists real numbers \( a, b \) and \( c \) satisfying \( 0 < a < 1, b \in (0, 1/2), c \in (0, 1/2) \) such that for each \( x, y \in X \), at least one of the following conditions holds:

\( (z_1) \) \( d(Tx, Ty) \leq ad(x, y), \)

\( (z_2) \) \( d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \)

\( (z_3) \) \( d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]. \)

Then \( T \) has a unique fixed point \( p \) and the Picard iteration \( \{x_n\} \) defined by
\[
x_{n+1} = Tx_n, \quad n \geq 1
\]
converges to \( p \) for any arbitrary \( x_1 \in X \).

**Remark 1.**

(1) A mapping \( T \) satisfying the conditions \((z_1), (z_2)\) and \((z_3)\) in the above theorem is called Zamfirescu operator or \( z \)-operator.

(2) When \( T \) is a \( z \)-operator, then we can get
\[
d(Tx, Ty) \leq \delta d(x, y) + 2\delta d(x, Tx), \quad \text{for all } x, y \in X,
\]
where \( \delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\} \) (cf. [15, 16, 17]).

In 2005, Berinde[18] introduced a new class of operators satisfying
\[
d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx),
\]
for all \( x, y \in X, 0 < \delta < 1 \) and \( L \geq 0 \). He proved that this class is wider than the class of \( z \)-operator and used the Ishikawa iteration process to approximate the fixed points of this class of operators in a normed space. See also[19].

Very recently, Khan[20] used Ishikawa iteration process with errors to approximate common fixed points of two mappings in a normed space. He proved the following main theorem.
Theorem 1.2 ([20]) Let $C$ be a nonempty closed bounded convex subset of a normed space $E$. Let $S, T : C \to C$ be two operators satisfying
\[
\|Tx - Ty\| \leq \delta \|x - y\| + L\|x - Tx\|, \text{ for all } x, y \in C,
\]
and
\[
\|Sx - Sy\| \leq \delta \|x - y\| + L\|x - Sx\| \text{ for all } x, y \in C,
\]
where $0 < \delta < 1, L \geq 0$. Let $\{x_n\}$ be defined by the following iterative process
\[
\begin{align*}
x_1 & \in C, \\
x_{n+1} &= a_nSy_n + b_nx_n + c_nu_n, \\
y_n &= a'_nTx_n + b'_nx_n + c'_nv_n, \quad n \geq 1.
\end{align*}
\]
If $F(S) \cap F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and $\lim_{n \to \infty} c'_n = 0$, then $\{x_n\}$ converges strongly to a common fixed point of $S$ and $T$.

Takahashi [21] introduced a notion of convex metric spaces as follows.

Definition 1.3 Let $(X, d)$ be a metric space and $I = [0, 1]$. A mapping $W : X^2 \times I \to X$ is said to be convex structure on $X$, if for any $(x, y, \lambda) \in X^2 \times I$ and $u \in X$ the following inequality holds
\[
d(W(x, y, \lambda), u) \leq \lambda d(x, u) + (1 - \lambda)d(y, u).
\]
If $(X, d)$ is a metric space with a convex structure $W$, then $(X, d)$ is called a convex metric space. Moreover, a nonempty subset $K$ of $X$ is said to be convex if $W(x, y, \lambda) \in K$, for all $(x, y, \lambda) \in K^2 \times I$.

Remark 2. (i) For all normed spaces and their convex subsets are convex metric spaces. But there also many examples of convex metric spaces which are not embedded in any normed space (see e.g., [22]).

(ii) For a Banach space, or any convex subset of it, is a convex metric space such that $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

As a generalization of Definition 1.3, we have the following definition.

Definition 1.4 ([23, 24]) Let $(X, d)$ be a metric space and $I = [0, 1]$. A mapping $W : X^3 \times I^3 \to X$ is said to be convex structure on $X$, if for any $(x, y, z, a, b, c) \in X^3 \times I^3$ and $u \in X$, the following inequality holds:
\[
d(W(x, y, z, a, b, c), u) \leq ad(x, u) + bd(y, u) + cd(z, u),
\]
where $a, b, c$ are real numbers in $[0, 1]$ with $a + b + c = 1$. If $(X, d)$ is a metric space with a convex structure on $W$, then $(X, d)$ is called a generalized convex metric space. Moreover, a nonempty subset $K$ of $X$ is said to be generalized convex if $W(x, y, z, a, b, c) \in K$ for all $(x, y, z, a, b, c) \in K^3 \times I^3$. 

Remark 3. (i) It is easy to see that every generalized convex metric space is a convex metric space.

(ii) For a Banach space, or any generalized convex subset of it, is a generalized convex metric space with \( W(x, y, z, a, b, c) = ax + by + cz \).

Definition 1.5 Let \( C \) be a nonempty closed convex subset of a generalized convex metric space \( X \). \( T_1, T_2 : C \rightarrow C \) be a pair of mappings. The sequence \( \{x_n\} \) defined by

\[
\begin{align*}
    x_1 & \in C, \\
    y_n & = W(x_n, T_2 x_n, v_n, a'_n, b'_n, c'_n), \\
    x_{n+1} & = W(x_n, T_1 y_n, u_n, a_n, b_n, c_n), n \geq 1,
\end{align*}
\]

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0, 1]\) with \(a_n + b_n + c_n = a'_n + b'_n + c'_n = 1\) and \(\{u_n\}, \{v_n\} \) are bounded sequences in \( C \), is called Ishikawa type iteration process with errors for a pair of mappings.

Remark 4. The iteration process (1.2) includes some well-known iteration processes \([2, 3, 6]\) as its special case. In particular, if \( X \) is a Banach space, then iteration process (1.2) reduces to

\[
\begin{align*}
    x_1 & \in C, \\
    y_n & = a'_n x_n + b'_n T_2 x_n + c'_n v_n, \\
    x_{n+1} & = a_n x_n + b_n T_1 y_n + c_n u_n, n \geq 1.
\end{align*}
\]

This type of iteration process has been studied by many authors to find common fixed points of two mappings in Banach spaces (see, for example, \([25, 26, 27, 28]\)). Takahashi [29] showed that approximating common fixed points of two mappings has its own important as it has a direct link with the minimization problem.

The purpose of this paper is to study the convergence of Ishikawa type iteration process with errors to a common fixed point of a pair of mappings in complete generalized convex metric spaces. Our results show that the boundedness requirement imposed on the subset of \( C \) in Theorem 1.2 of Khan [20] can be dropped. Our results also extend and generalize the corresponding results of Berinde [17] and others.

In the sequel, we shall need the following lemma.

Lemma 1.6 ([5]) Let \( r_n, s_n, t_n \) and \( k_n \) be nonnegative real sequences satisfying the following inequality:

\[ r_{n+1} \leq (1 - s_n) r_n + t_n + k_n, \quad \forall n \geq 1, \]

where \( s_n \in [0, 1], \sum s_n = +\infty, t_n = o(s_n) \) and \( \sum k_n < +\infty \). Then \( r_n \rightarrow 0 \) as \( n \rightarrow \infty \).
2 Main Results

Now, we give our main results in complete generalized convex metric spaces.

**Theorem 2.1** Let $C$ be a nonempty closed convex subset of a complete generalized convex metric space $X$. Let $T_1, T_2 : C \to C$ be a pair of mappings satisfying:

$$d(T_i x, T_i y) \leq \delta_i d(x, y) + L_i d(T_i x, x),$$

for all $x, y \in C$, where $\delta_i \in (0, 1), L_i \geq 0$. Let $\{x_n\}$ be defined by (1.2) and satisfying:

$$\sum_{n=1}^{\infty} b_n = \infty, \sum_{n=1}^{\infty} c_n < \infty \text{ and } \lim_{n \to \infty} c'_n = 0.$$

If $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of $T_1$ and $T_2$.

**Proof.** Let $\delta = \max\{\delta_1, \delta_2\}, L = \max\{L_1, L_2\}$, then $\delta \in (0, 1), L \geq 0$ and $T_1, T_2 : C \to C$ satisfying

$$d(T_i x, T_i y) \leq \delta d(x, y) + L d(T_i x, x),$$

for all $x, y \in C, i = 1, 2$. (2.1)

Let $p \in F(T_1) \cap F(T_2)$, using iteration method (1.2), we have

$$d(x_{n+1}, p) = d(W(x_n, T_1 y_n, u_n, a_n, b_n, c_n), p) \leq a_n d(x_n, p) + b_n d(T_1 y_n, p) + c_n d(u_n, p).$$

If $i = 1, x = p$ and $y = y_n$ in (2.1), we get

$$d(T_1 y_n, p) \leq \delta d(y_n, p).$$

Thus,

$$d(x_{n+1}, p) \leq a_n d(x_n, p) + b_n \delta d(y_n, p) + c_n d(u_n, p).$$

(2.2)

By (1.2), we obtain

$$d(y_n, p) = d(W(x_n, T_2 x_n, v_n, a'_n, b'_n, c'_n), p) \leq a'_n d(x_n, p) + b'_n d(T_2 x_n, p) + c'_n d(v_n, p),$$

If $i = 2, x = p$ and $y = x_n$ in (2.1), we have

$$d(T_2 x_n, p) \leq \delta d(x_n, p).$$

Hence,

$$d(y_n, p) \leq a'_n d(x_n, p) + b'_n \delta d(x_n, p) + c'_n d(v_n, p).$$

(2.3)
Substituting (2.3) into (2.2), we have

\[
d(x_{n+1}, p) \leq a_n d(x_n, p) + a'_n b_n \delta d(x_n, p) + b_n b'_n \delta^2 d(x_n, p) \\
+ b_n c'_n \delta d(v_n, p) + c_n d(u_n, p) \\
= (1 - b_n - c_n + b_n \delta (1 - b'_n - c'_n) + b_n b'_n \delta^2) d(x_n, p) \\
+ b_n c'_n \delta d(v_n, p) + c_n d(u_n, p) \\
= (1 - (1 - \delta) b_n - c_n - b_n \delta c'_n - b_n b'_n \delta (1 - \delta)) d(x_n, p) \\
+ b_n c'_n \delta d(v_n, p) + c_n d(u_n, p) \\
\leq (1 - (1 - \delta) b_n) d(x_n, p) + b_n c'_n \delta d(v_n, p) + c_n d(u_n, p).
\]

Since \(1 - \delta > 0\) and notice the condition of \(\sum_{n=1}^{\infty} b_n = \infty, \lim_{n \to \infty} c'_n = 0\) and \(\sum_{n=1}^{\infty} c_n < \infty\), by Lemma 1.6, we know that \(d(x_n, p) \to 0\) as \(n \to \infty\). Hence, \(\{x_n\}\) converges strongly to a common fixed point of \(T_1\) and \(T_2\).

By Remark 3, we have the following strong convergence result in Banach spaces respectively.

**Theorem 2.2** Let \(C\) be a nonempty closed convex subset of a Banach space \(X\), and \(T_1, T_2 : C \to C\) be a pair of mappings satisfying

\[
\|T_ix - T_iy\| \leq \delta_i \|x - y\| + L_i \|x - T_ix\|, \text{ for all } x, y \in C,
\]

where \(\delta_i \in (0, 1), L_i \geq 0, i = 1, 2\). For any \(x_1 \in C\), \(\{x_n\}\) is defined by

\[
x_{n+1} = a_n x_n + b_n T_1 y_n + c_n u_n, \\
y_n = a'_n x_n + b'_n T_2 x_n + c'_n v_n, n \geq 1,
\]

where \(\{u_n\}, \{v_n\}\) are bounded sequences in \(C\) and \(\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}\) are sequences in \([0, 1]\) satisfying

(i) \(a_n + b_n + c_n = a'_n + b'_n + c'_n = 1\);

(ii) \(\sum_{n=1}^{\infty} b_n = \infty, \sum_{n=1}^{\infty} c_n < \infty\) and \(\lim_{n \to \infty} c'_n = 0\).

If \(F(T_1) \cap F(T_2) \neq \emptyset\), then \(\{x_n\}\) converges strongly to a common fixed point of \(T_1\) and \(T_2\).

**Remark 5.** (i) Theorem 2.2 discarded the boundedness requirement on the subset of \(C\) in Theorem 1.2 of Khan[20].

(ii) Theorem 2.2 extended Theorem 2 of Berinde[17] to a more general class of two mappings and more general iteration process with errors.
References


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