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A Fixed Point Theorem for Four Weakly Compatible Mappings in Fuzzy Metric Spaces

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Abstract

The aim of this paper is to consider a new approach for obtaining common fixed point theorems in fuzzy metric space for weakly compatible maps by using the concept of (E. A.) property. We are proving the result of Singh and Chouhan [9] by using this new approach.

Keywords: Fuzzy metric space, compatible maps, weak compatible maps, common fixed point, E.A property

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1. Introduction

Zadeh's [10] introduction of the notion of fuzzy set laid the foundation of fuzzy mathematics in this paper we deal with the fuzzy metric space defined by Kramosil and Michalek [6] and modified by George and Veeramani [3]. In 1986 Jungck [4] introduced the notion of compatible maps for a pair of self mappings. Later on compatible maps were introduced by Mishra et. al [7] in fuzzy metric space Jungck and Rhoades [5] termed a pair of self maps to be coincidently commuting or equivalently weak compatible if they commute at their coincidence points. Aamri and El Moutawakil [1] generalized the concept of non compatibility by defining the notion of E.A. property and proved common fixed point theorems under strict contractive conditions. In this paper we extend this concept to fuzzy metric space and establish the existence of common fixed points for a pair of self maps.

Before we start we give some preliminaries.

Definition1.1[10] Let X be any set. A fuzzy set A in X is a function with domain X and values in [0,1].

Definitions 1.2[8] A binary operation $*\ [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions :

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a, \ \forall \ a \in [0,1]$
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition1.3[6] The triplet (X, M, $*$) is said to be fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set on $X \times X \times [0,\infty] \to [0,1]$ satisfying the following conditions :

for all x, y, z $\in$ X and s, t > 0.

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0,\infty] \to [0,1]$ is left continuous,
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that M(x, y, t) can be considered as the degree of nearness between x and y with respect to t. We identify x = y with M(x, y, t) = 1 for all t > 0. The following example shows that every metric space induces a fuzzy metric space.

Example1.4[3] Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x,y,t) = \frac{t}{t + d(x,y)}$ for all x, y $\in$ X and all t > 0. Then (X, M, $*$) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Lemma 1.5[2] For all x, y $\in$ X, M(x, y, .) is a non-decreasing function.

Lemma 1.6[7] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, $*$) with $t*t > t$ for all $t\in[0,1]$ and condition (FM-6). If there exists a number k $\in$ (0,1) such that

$$M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t)$$

for all t > 0 and n = 1, 2 . . . then $\{x_n\}$ is a Cauchy sequence in X

Lemma 1.7[7] If for all x, y ∈ X, t > 0 with positive number k ∈ (0,1) and M(x, y, kt) ≥ M(x, y, t) then x = y.

Definition1.8[9] Let S and T be two self maps of a fuzzy metric space (X,M,∗), S and T are Said to be compatible if $M(STx_n, TSx_n, t) \to 1$ as n→ ∞, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to z$ as n→ ∞, for some z ∈ X.

Definition1.9[5] Two self maps S and T of Fuzzy metric space (X, M, ∗) are said to be weakly compatible if they commute at their coincidence point, i.e. STu = TSu whenever Su = Tu, u ∈ X .

The concept of weak compatibility is most general among all the commutativity concepts, clearly each pair of compatible self maps is weakly compatible but the converse is not true always.

Definition1.10 Let S and T be two self maps of a fuzzy metric space (X, M, ∗) we say that S and T satisfy E.A property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{t\to\infty} Sx_n = \lim_{t\to\infty} Tx_n = z$ for some z ∈ X.

Example 1.11 Let X = [0,∞] and $M(x, y, t) = \frac{t}{t+|x-y|}$.

Defines S,T : X→ $[0,\infty]$ by $Tx = \frac{x}{7}$ and $Sx = \frac{2x}{7}$ for all x ∈ X.

Then $\lim_{t\to\infty} Sx_n = \lim_{t\to\infty} Tx_n = 0$, where $x_n = 1/n$.

2. Main Result

In the theorem 3.1[9] we involves a function F: [0,1] → [0,1] satisfy the following conditions:

(a) F is increasing on [0,1]

(b) F(t) > t for each t > 1 and F(1) = 1.

Theorem 2.1 Let A, B, T and S be self mappings of a fuzzy metric space (X,M,∗) such that

(1) M(Ax, By, t), ≥ F max{M(Sx, Ty, t),M(Ax, Sx, t), M(By, Ty, t), M(By, Sx, 2t), M(Ax, Ty, t)}, ∀ x≠y ∈X, t > 0.

(2) (A, S) and (B, T) are weakly compatible,

(3) (A, S) or (B, T) satisfies E.A property,

(4) Ax ⊂ Tx and Bx ⊂ Sx,

If any of the ranges of A, B, T and S is complete subspace of X, then A, B, T and S have a unique common fixed point in X.

Proof: Suppose that (B,T) satisfies the E.A property. Then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$ for some $p \in X$.
Since $Bx \subset Sx$ there exists a sequence $\{y_n\}$ in X such that
$Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = p$.
Now we will show that $\lim_{n\to\infty} Ay_n = p$
By condition (1) of the theorem
$M(Ay_n, Bx_n, t) \geq F\ \max\{M(Sy_n, Tx_n, t), M(Ay_n, Sy_n, t), M(Bx_n, Tx_n, t), M(Bx_n, Sy_n, 2t), M(Ay_n, Tx_n, t)\}$

$$\lim_{n\to\infty} M(Ay_n, Bx_n, t) \geq F\ \max\{1, M(Ay_n, p, t), 1, 1, M(Ay_n, p, t)\}$$
$$= F(1)$$
$$= 1$$

Therefore $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = p$.
Suppose that SX is a complete subspace of X. Then there exists some point u in X such that $p = Su$. Therefore we have
$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = p = Su$.
Now, we shall show that $Au = Su$. From (1) we have
$M(Au, Bx_n, t) \geq F\ \max\{M(Su, Tx_n, t), M(Au, Su, t), M(Bx_n, Tx_n, t), M(Bx_n, Su, 2t), M(Au, Tx_n, t)\}$
Taking limit $n\to\infty$

$$M(Au, Su, t)\ ) \geq F\ \max\{M(Su, Su, t), M(Au, Su, t), M(Su, Su, t), M(Su, Su, 2t), M(Au, Su, t)\}.$$
$$= F\ \max\{1, M(Au, Su, t), 1, 1, M(Au, Su, t)\}$$
$$= F(1) = 1$$

$\Rightarrow Au = Su$.

Case I - As the pair of mappings (A, S) is weakly compatible, so $ASu = SAu$ and then
$AAu = ASu = SAu = SSu$.
Since $AX \subset TX$, there exists $v \in X$ such that $Au = Tv$.
Now we shall show that $Tv = Bv$.

$$M(Au, Bv, t) \geq F\ \max\{M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), M(Bv, Su, 2t), M(Au, Tv, t)\}$$
$$M(Au, Bv, t) \geq F\ \max\{M(Tv, Tv, t), M(Tv, Tv, t), M(Bv, Tv, t), M(Bv, Tv, 2t), M(Tv, Tv, t)\}$$
$$= F(1) = 1$$

$\Rightarrow Au = Bv$
Therfore $Tv = Bv$
Hence we have $Au = Su = Tv = Bv$.

Case II- As the pair of mappings (B, T) is weakly compatible, so $BTv = TBv$
and hence $BBv = BTv = TBv = TTv$
Finally we shall show that Au is the common fixed point of A, B, T and S.
Put $x = u$, $y = Au$ in (1)

$$\begin{aligned} M(Au, AAu, t) &= M(AAu, Bv, t) \\ &\geq F \max\{M(SAu, Tv, t), M(AAu, SAu, t), M(Bv, Tv, t), \\ &\qquad M(Bv, SAu, 2t), M(AAu, Tv, t)\} \\ &= F \max\{M(AAu, Au, t), M(AAu, AAu, t), M(Bv, Tv, t), \\ &\qquad M(Au, AAu, 2t), M(AAu, Au, t)\} \\ &= F(1) = 1 \end{aligned}$$

$\Rightarrow Au = AAu$

Therefore $Au = AAu = SAu$

Hence Au is the common fixed point of A and S.

Similarly we can prove that Bv is the common fixed point of B and T i.e. $BBv = TBv = Bv$ since $Au = Bv$, therefore $BAu = TAu = Au$.

Hence Au is the common fixed point of A, B, T and S.

The cases in which AX or BX is a complete subspace of X are similar to the cases in which TX or SX, respectively is complete subspace of X, since $AX \subset TX$ and $BX \subset SX$.

Uniqueness- let u and v be two common fixed points of A, B, T and S. Then

$$\begin{aligned} M(u, v, t) &= M(Au, Bv, t) \\ &\geq F \max\{M(Su, Tv, t), M(Au, Su, t), M(Bv, Tv, t), M(Bv, Su, 2t), \\ &\qquad M(Au, Tv, t)\} \\ &= F(1) = 1 \end{aligned}$$

Which implies $u = v$

Therefore A, B, T and S have unique common fixed point.

Hence completes the proof.

Corollary 2.2 Let A, B and S be self mappings of a fuzzy metric space $(X,M,*)$ such that

(1) $M(Ax, By, t), \geq F \max\{M(Sx, Sy, t), M(Ax, Sx, t), M(By, Sy, t), M(By, Sx, 2t), M(Ax, Sy, t)\}$
$\forall\ x \neq y \in X, t > 0.$

(2) (A, S) and (B, S) are weakly compatible,

(3) (A, S) or (B, S) satisfies E.A property,

(4) $AX \subset SX$ and $BX \subset SX$.

If any of the ranges of A, B and S is complete subspace of X, then A, B, and S have a unique common fixed point.

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