

Global Stability of High-Dimensional System in Epidemiology with Nonlinear Incidence Rates

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Abstract

The model *SEIR* with nonlinear incidence rates in epidemiology is further studied. In this paper, the global analysis of *SEIR* is considered, including the orbital stability of periodic orbits, Hopf bifurcation, steady switch phenomenon and so on.

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1. Introduction

The *SEIR* model with nonlinear incidence rates is described as the following system of differential equations:

$$\begin{aligned} S' &= -\lambda I^p S^q + b - \mu S, \\ E' &= \lambda I^p S^q - (\varepsilon + \mu)E, \\ I' &= \varepsilon E - (\gamma + \mu)I, \\ R' &= \gamma I - \mu R. \end{aligned} \tag{1.1}$$

Here $p, q, \varepsilon, \lambda, \gamma, b$ are positive parameters, the population is partitioned into four compartments that are susceptible, exposed, infectious, and recovered (with permanent immunity), with sizes denoted by S, E, I, R , respectively. μ is a non-negative constant and represents the death, b is the rate of birth, ε is the rate constant at which the exposed individuals become infective, γ is the rate for recovery.

It is traditionally postulated that the spread of an infection occurs according to the principle of mass action and associated with the nonlinear incidence rates. In this note, we consider global behavior of a $SEIR$ model with the incidence rate of the form $\lambda I^p S^q$ for the peculiar case $p > 1$.

In recent years models with this incidence rate were considered by several authors, for example Liu et al^[1,2]. Hethcote et al^[3]. Hethcote and Van den Driessche^[4]. Derrick and Van den Driessche^[5,6]. Glendinning and Perry^[7]. Lizana and Rivero^[8] and others. With very few exceptions, these authors focused on the local properties and bifurcation of equilibrium states.

2. Statement of the main result

Throughout this paper, we assume that $p > 1$.

The feasible region for (1.1) is R_+^4 , the positive orthant of R^4 . Adding all the equations in (1.1), we have $\Gamma = \{(S, E, I, R) \in R_+^4 : S + E + I + R = 1\}$ that is positive invariant, on the simplex $\Gamma : R(t) = 1 - S(t) - E(t) - I(t)$. We consider the birth equal to the death. Thus (1.1) reduces to the following 3-dimensional system:

$$\begin{aligned} S' &= -\lambda I^p S^q + \mu - \mu S, \\ E' &= \lambda I^p S^q - (\varepsilon + \mu)E, \\ I' &= \varepsilon E - (\gamma + \mu)I. \end{aligned} \quad (1.2)$$

Let $\tau = (\gamma + \mu)t$, $\alpha = \frac{\mu}{\gamma + \mu}$, $\beta = \frac{\varepsilon}{\gamma + \mu}$, $a = \frac{\lambda}{\gamma + \mu}$ and $p = 2$, $q = 1$. Then the system (1.2) can be changed into

$$\begin{aligned} S' &= -aI^2 S + \alpha - \alpha S, \\ E' &= aI^2 S - (\alpha + \beta)E, \\ I' &= \beta E - I. \end{aligned} \quad (2.1)$$

The dynamic behavior of (1.1) on Γ is equivalent to that of (2.1). Therefore, in the rest of the paper we will study the system (2.1) in the region $T = \{(S, E, I) : 0 \leq S, E, I \leq 1, S + E + I \leq 1\}$. The more detailed study can be found in [9].

Theorem 2.1 *If $p = 2$, $q = 1$ and $\bar{\sigma} = \frac{\alpha\alpha\beta^2}{4(\alpha+\beta)^2} < 1$, there is only a disease-free equilibrium $P_0(1, 0, 0)$ which is globally asymptotically stable in the interior of T .*

Proof A Lyapunov function

$$V(S, E, I) = \left(S - S^* - S^* \ln \frac{S}{S^*} \right) + \frac{\alpha + \beta}{\beta} I \left(1 + \frac{1}{p-1} \left(\frac{I^{*p}}{I} \right) \right) + (E - E^* \ln E)$$

is defined and continuous for all $S, E, I > 0$. Because $S^*, E^*, I^* = (1, 0, 0)$, so it becomes $V(S, E, I) = (S - S^* - S^* \ln \frac{S}{S^*}) + \frac{\alpha + \beta}{\beta} I + E$, and it satisfies $\frac{\partial V}{\partial S} = 1 - \frac{1}{S}$, $\frac{\partial V}{\partial E} = 1$, $\frac{\partial V}{\partial I} = \frac{\alpha + \beta}{\beta}$ and $V(S^*, E^*, I^*) = 0$. In the case of system (2.1), we can get

$$\begin{aligned}
 V' &= \left(1 - \frac{1}{S}\right) (-aI^2S + \alpha - \alpha S) + [aI^2S - (\alpha + \beta)E] + \frac{\alpha + \beta}{\beta}(\beta E - I) \\
 &= aI^2 - \frac{\alpha + \beta}{\beta}I + \alpha(1 - S) \left(1 - \frac{1}{S}\right).
 \end{aligned}$$

Let $f(I) = aI^2 - \frac{\alpha + \beta}{\beta}I$, because of $R_0 = \frac{\lambda\varepsilon}{(\gamma + \mu)(\varepsilon + \mu) - \frac{\alpha\beta}{\alpha + \beta}} < 1$, we will get $f(I) \leq 1$ in $\left(0, \frac{\alpha + \beta}{\alpha\beta}\right) = \left(0, \frac{1}{R_0}\right)$ and it will lead to $V' \leq 0$. Disease-free equilibrium $P_0(1, 0, 0)$ is globally asymptotically stable. The theorem is proved.

Remark 2.1 If $p = 1, q = 1$, we can let

$$V(S, E, I) = A \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + B \left(I - I^* - I^* \ln \frac{I}{I^*}\right) + C(E - E^* - E^* \ln E).$$

Remark 2.2 If $\bar{\sigma} = \frac{a\alpha\beta^2}{4(\alpha + \beta)^2} = 1$, there may yield periodic orbits.

If $\bar{\sigma} = \frac{a\alpha\beta^2}{4(\alpha + \beta)^2} = 1$, and $2\alpha(\alpha + \beta + 1) - (\alpha + \beta) < 0$, there are a zero eigenvalue, a negative and a positive eigenvalue. The endemic- equilibrium P^* is non-stable. We discuss the existence of special periodic orbits under the positive invariant $B^1 = \left\{(S, E, I) : 0 \leq S \leq 1, 0 \leq E \leq \frac{\alpha\beta^2}{\alpha + \beta}, 0 \leq I \leq \beta, \beta \leq 1\right\}$, divide the region into eight sub-regions as following:

$$\begin{aligned}
 B_1 &= \{(S, E, I) \in B^1 : S < S^*, E < E^*, I < I^*\}; \\
 B_2 &= \{(S, E, I) \in B^1 : S > S^*, E < E^*, I < I^*\}; \\
 B_3 &= \{(S, E, I) \in B^1 : S > S^*, E > E^*, I < I^*\}; \\
 B_4 &= \{(S, E, I) \in B^1 : S > S^*, E > E^*, I > I^*\}; \\
 B_5 &= \{(S, E, I) \in B^1 : S < S^*, E > E^*, I > I^*\}; \\
 B_6 &= \{(S, E, I) \in B^1 : S < S^*, E < E^*, I > I^*\}; \\
 B_7 &= \{(S, E, I) \in B^1 : S > S^*, E < E^*, I > I^*\}; \\
 B_8 &= \{(S, E, I) \in B^1 : S < S^*, E > E^*, I < I^*\}.
 \end{aligned}$$

Analysis the trajectory in every sub-region in the following figure, we will get that the trajectory are spiring following $B_2 \rightarrow B_3 \rightarrow B_4 \rightarrow B_5 \rightarrow B_6 \rightarrow B_1 \rightarrow B_2$. Besides, there may be periodic orbits, because some $t_0 > 0$ returns to $F \subset (B_1 \cap B_2)$, we can define a map: $\Phi : F \rightarrow F$. Under theorem *Brouwer*, there will exists $\hat{q} \in F$, it will make $\Phi(\hat{q}) = \hat{q}$.

Theorem 2.2 A sufficient condition for a periodic orbit $\gamma = \{P(t) : 0 \leq t \leq \omega$ of (2.1) to be asymptotically orbitally stable with asymptotic phase is that the linear

system:

$$Z'(t) = \frac{\partial f^{[2]}}{\partial x}(P(t))Z(t), \quad (2.2)$$

be asymptotically stable.

Equation (2.2) is called the second compound equation of $y' = \frac{\partial f}{\partial x}(x(t, x_0))y(t)$ and $\frac{\partial f^{[2]}}{\partial x}$ is the second compound matrix of the Jacobian matrix $\frac{\partial f}{\partial x}$ of f .

Theorem 2.3 *The trajectory of any nonconstant periodic solution to (2.1), if it exists, is asymptotically orbitally stable with asymptotic phase.*

Proof We can write the linear system with respect to the solution $(S(t), E(t), I(t))$ of (2.1) as the following 3×3 system:

$$\begin{aligned} X' &= -(aI^2 + 2\alpha + \beta)X + 2aIS(Y + Z), \\ Y' &= \beta X - (aI^2 + \alpha + 1)Y, \\ Z' &= aI^2Y - (\alpha + \beta + 1)Z. \end{aligned} \quad (2.3.1)$$

To show the asymptotic stability of the system (2.3.1), we consider the following function:

$$V(X, Y, Z; S, E, I) = \sup \left\{ |X|, \frac{E}{I}(|Y| + |Z|) \right\}.$$

Suppose that the solution $(S(t), E(t), I(t))$ is periodic of least period $\omega > 0$, as to the orbit γ remains at a positive distance from the boundary of T , there exists constant such that

$$V(X, Y, Z; S, E, I) \geq \sup \{|X|, |Y|, |Z|\}, \quad (2.3.2)$$

for all $(X, Y, Z) \in R_+^3$ and $(S, E, I) \in \gamma$. The right-hand derivative of $V(t)$ exists and the direct calculation yields

$$\begin{aligned} D_+|X(t)| &\leq -(aI^2 + 2\alpha + \beta)|X| + 2aIS(|Y| + |Z|) \\ &= -(aI^2 + 2\alpha + \beta)|X| + \frac{2aI^2S}{E} \left\{ \frac{E}{I}(|Y| + |Z|) \right\}, \end{aligned} \quad (2.3.3)$$

and

$$\begin{aligned} D_+|Y(t)| &\leq \beta|X| - (aI^2 + \alpha + 1)|Y|, \\ D_+|Z(t)| &\leq aI^2|Y| - (a + \beta + 1)|Z|, \end{aligned}$$

and thus

$$\begin{aligned}
D_+ \frac{E}{I} (|Y| + |Z|) &= \left(\frac{E'}{I} - \frac{I'}{E} \right) \frac{E}{I} (|Y| + |Z|) + \frac{E}{I} D_+ (|Y| + |Z|) \\
&\leq \left(\frac{E'}{I} - \frac{I'}{E} \right) \frac{E}{I} (|Y| + |Z|) + \frac{E}{I} (\beta |X| - (\alpha + 1) |Y| + (\alpha + \beta + 1) |Z|) \\
&\leq \frac{\beta E}{I} |X| + \left(\frac{E'}{I} - \frac{I'}{E} - \alpha - 1 \right) \frac{E}{I} (|Y| + |Z|).
\end{aligned} \tag{2.3.4}$$

We claim that (2.3.3) and (2.3.4) lead to

$$D_+ V(t) \leq \sup\{g_1(t), g_2(t)\} V(t),$$

where

$$g_1(t) = -(\alpha I^2 + 2\alpha + \beta) + \frac{2\alpha I^2 S}{E}, \quad g_2(t) = \frac{\beta E}{I} + \frac{E'}{E} - \frac{I'}{I} - \alpha - 1.$$

Using (2.1), we find that

$$\frac{I'}{I} + 1 = \frac{E}{I}, \quad \frac{E'}{E} + \alpha + \beta = \frac{aI^2 S}{E}.$$

We can get

$$D_+ V(t) \leq \sup \left\{ \frac{E'}{E} - \alpha, \frac{2E'}{E} + \beta - aI^2 \right\} V(t),$$

because $(S(0), E(0), I(0)) \in T^0$, there exists suitable parameter β and α that satisfies $\beta - \alpha t^2 < 0$, and thus

$$\int_0^\omega \sup\{g_1(t), g_2(t)\} dt < 0,$$

which, together with (2.3.4), implies that $V(t) \rightarrow 0$ as $t \rightarrow \infty$, and in turn that $(X(t), Y(t), Z(t)) \rightarrow 0$ as $t \rightarrow \infty$ by (2.3.2). As a result, the linear system (2.3.1) is asymptotically stable and the periodic solution $(S(t), E(t), I(t))$ is asymptotically orbitally stable with asymptotic phase.

Theorem 2.4 *If $\bar{\sigma} = \frac{a\alpha\beta^2}{4(\alpha+\beta)^2} > 1$, the system (2.1) must yield Hopf bifurcation, steady switch phenomenon.*

Proof There are three equilibria to (2.1) when parameters satisfy $\bar{\sigma} = \frac{a\alpha\beta^2}{4(\alpha+\beta)^2} > 1$: the disease -free equilibrium $P_0(1, 0, 0)$, two endemic equilibrium P_1 and P_2 . P_0 is locally asymptotically stable, P_2 is unstable. We can show in the following table about P_1 :

3. Discussion

It is significative to research higher-dimensional periodic orbits further, and whether there is only a periodic orbit is not settled.

parameter	eigenvalue	eigenvalue	eigenvalue	character
$a = 500$	$\lambda_1 = -2.4216$	$\lambda_2 = 0.3698$	$\lambda_3 = 0.0168$	unstable
$a = 1000$	$\lambda_1 = -2.4371$	$\lambda_2 = 0.2329$	$\lambda_3 = 0.1175$	unstable
$a = 2000$	$\lambda_1 = -2.4676$	$\lambda_2 = 0.1411 + 0.2171i$	$\lambda_3 = 0.1411 - 0.2171i$	unstable
$a = 6362$	$\lambda_1 = -2.6135$	$\lambda_2 = 0.0001 + 0.4765i$	$\lambda_3 = 0.0001 - 0.4765i$	unstable
$a = 6363$	$\lambda_1 = -2.6135$	$\lambda_2 = 0.0000 + 0.4765i$	$\lambda_3 = 0.0000 - 0.4765i$	
$a = 6364$	$\lambda_1 = -2.6136$	$\lambda_2 = 0.0000 + 0.4766i$	$\lambda_3 = -0.0000 - 0.4766i$	
$a = 6365$	$\lambda_1 = -2.6136$	$\lambda_2 = 0.0001 + 0.4766i$	$\lambda_3 = -0.0001 - 0.4766i$	
$a = 6366$	$\lambda_1 = -2.6136$	$\lambda_2 = 0.0001 + 0.4766i$	$\lambda_3 = -0.0001 - 0.4766i$	unstable
$a = 7000$	$\lambda_1 = -2.6368$	$\lambda_2 = 0.0195 + 0.4984i$	$\lambda_3 = -0.0195 - 0.4984i$	unstable

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