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# A Semi-Symmetric Non-Metric $\phi$-Connection in a Lorentzian Para-Sasakian Manifold

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**Abstract**

In this paper we introduce a semi-symmetric non-metric $\phi$-connection in a Lorentzian Para-Sasakian (LP-Sasakian) manifold . Some geometrical results related to this connection are obtained.

**Keywords:** Semi-symmetric non-metric $\phi$-connection, Lorentzian Para-Sasakian manifold, conformal, con-harmonic and concircular curvature tensor.

## 1 Introduction.

In 1977, K. Yano and Tyuti Imai [1], studied semi-symmetric metric $\phi$-connections in a Sasakian manifold. In 2001, V. K. Jaiswal, R. H. Ojha and B. Prasad [2], introduced a semi-symmetric metric $\phi$-connection in LSP-Sasakian manifold.

In this paper we introduce a semi-symmetric non-metric $\phi$- connection in LP-Sasakian manifold and obtain its curvature tensor. At first it is shown that LP-Sasakian manifold admitting a semi-symmetric non-metric $\phi$-connection is closed with respect Riemannian connection $D$ if and only if it is closed with respect to semi-symmetric non-metric $\phi$-connection $\widetilde{B}$. Finally we prove that if an LP-Sasakian manifold $M_n$ admitting a semi-symmetric non-metric $\phi$-connection whose curvature is of some specified form then the conformal curvature tensor $V$ concides with the con-harmonic curvature tensor $L$ and con-circular curvature tensor $C$ concides with the Riemannian curvature tensor $R$ and for another specified form, it is conformally flat.

## 2 Preliminaries.

An $n$-dimensional differentiable manifold $M_n$ is an LP-Sasakian manifold if it admits a $(1, 1)$ tensor field $\phi$, a vector field $\xi$ and a 1-form $\eta$ and a Lorentzian metric $g$ which satisfy

$$\phi^2(X) = X + \eta(X)\xi, \tag{1}$$

$$\eta(\xi) = -1, \tag{2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

$$g(X,\xi) = \eta(X), \tag{4}$$

$$(D_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{5}$$

and

$$D_X\xi = \phi X, \tag{6}$$

for arbitrary vector fields $X$ and $Y$, where $D$ denotes covariant differentiation with respect to g [3].

In an LP-Sasakian manifold $M_n$ with structure $(\phi, \xi, \eta, g)$ following holds

$$(a)\phi\xi = 0, (b)\eta(\phi X) = 0, \tag{7}$$

$$Rank(\phi) = n - 1. \tag{8}$$

If, we put

$$F(X,Y) = g(\phi X, Y), \tag{9}$$

then the tensor field $F$ is symmetric $(0,2)$ tensor field and it can be easily seen that

$$\begin{aligned} F(X,Y) = F(Y,X) &= F(\phi X, \phi Y) \\ &= g(\phi X, Y) = g(\phi Y, X), \end{aligned} \tag{10}$$

and

$$F(X,Y) = (D_X\eta)(Y). \tag{11}$$

# 3 Semi-symmetric non-metric $\phi$-connection in an LP-Sasakian manifold.

Let $D$ be the Riemannian connection in $M_n$ and $\widetilde{B}$ be an affine connection. Then $\widetilde{B}$ is said to be non-metric if it satisfies

$$\widetilde{B}_X g \neq 0. \tag{12}$$

In an LP-Sasakian manifold $M_n$ an affine connection $\widetilde{B}$ is called $\phi$-connection if it satisfies

$$\widetilde{B}_X \phi = 0. \tag{13}$$

Now we study non-metric $\phi$-connection having the torsion tensor of the form

$$S(X,Y) = 2\eta(Y)\phi X - 2\eta(X)\phi Y, \tag{14}$$

where $S$ is the torsion tensor of connection $\widetilde{B}$.

**Definition 3.1** *A linear connection $\widetilde{B}$ satisfying (12),(13),(14) is called a semi-symmetric non-metric $\phi$-connection.*

**Theorem 3.1** *In an LP-Sasakian manifold, the connection $\widetilde{B}$ defined by*

$$\widetilde{B}_X Y = D_X Y + \eta(Y)\phi X - F(X,Y)\xi - \eta(X)\phi Y, \tag{15}$$

*is a semi-symmetric non-metric $\phi$-connection whose metric is given by the relation*

$$(\widetilde{B}_X g)(Y,Z) = 2\eta(X)F(Y,Z). \tag{16}$$

**Proof.** If we put,

$$\widetilde{B}_X Y = D_X Y + H(X,Y), \tag{17}$$

where $H$ is the tensor field of the type (1,2) defined by

$$H(X,Y) = a\eta(Y)\phi X + bF(X,Y)\xi - \eta(X)\phi Y, \tag{18}$$

where $a$ and $b$ are constants. Then in LP-Sasakian manifold, we have

$$(\widetilde{B}_X \phi)(Y) = (D_X \phi)(Y) + H(X,\phi Y) - \phi H(X,Y). \tag{19}$$

Using (5) and (19),we have

$$(\widetilde{B}_X\phi)(Y) = H(X,\phi Y) - \phi H(X,Y) + g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{20}$$

In view (13) and (18), (20) gives

$$0 = (1-a)\eta(Y)X + (1+b)g(X,Y)\xi + (2+b-a)\eta(X)\eta(Y)\xi,$$

which gives

$$a = 1, b = -1.$$

Putting these values in (18) and using (17), we get (15).
Now, we have

$$\begin{aligned}(\widetilde{B}_X g)(Y,Z) =& X(g(Y,Z)) - g(\widetilde{B}_X Y, Z) - g(Y, \widetilde{B}_X Z)\\ =& X(g(Y,Z)) - g(D_X Y, Z) - \eta(Y)g(\phi X, Z)\\ &+ F(X,Y)g(Z,\xi) + \eta(X)g(\phi Y, Z)\\ &- g(Y, D_X Z) - \eta(Z)g(Y,\phi X)\\ &+ F(X,Z)g(Y,\xi) + \eta(X)g(Y,\phi Z)\\ =& 2\eta(X)F(Y,Z).\end{aligned}$$

Thus we have the statement.

# 4 Curvature Tensor of Semi-symmetric non-metric $\phi$-connection in an LP-Sasakian manifold.

Let $\widetilde{R}$ be curvature tensor of the connection $\widetilde{B}$ then

$$\widetilde{R}(X,Y,Z) = \widetilde{B}_X\widetilde{B}_Y Z - \widetilde{B}_Y\widetilde{B}_X Z - \widetilde{B}_{[X,Y]}Z. \tag{21}$$

From (15) and (21), we get

$$\begin{aligned}\widetilde{R}(X,Y,Z) =& \widetilde{B}_X(D_Y Z + \eta(Z)\phi Y - F(Y,Z)\xi - \eta(Y)\phi Z)\\ &- \widetilde{B}_Y(D_X Z + \eta(Z)\phi X - F(X,Z)\xi - \eta(X)\phi Z)\\ &- (D_{[X,Y]} + \eta(Z)\phi[X,Y] - F([X,Y],Z)\xi - \eta([X,Y])\phi Z)\\ =& R(X,Y,Z) + \eta(Z)[(D_X\phi)(Y) - (D_Y\phi)(X)]\\ &- F(Y,Z)\phi X + F(X,Z)\phi Y + \eta(Y)(D_X\phi)(Z) - \eta(X)(D_Y\phi)(Z)\\ &- \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X + \eta(Y)g(\phi X,\phi Z)\xi - \eta(X)g(\phi Y,\phi Z)\xi\\ =& R(X,Y,Z) + \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y\\ &- \eta(Y)g(X,Z)\xi + \eta(X)g(Y,Z)\xi + F(X,Z)\phi Y - F(Y,Z)\phi X.\end{aligned} \tag{22}$$

**Theorem 4.1** *If an LP-Sasakian manifold $M_n$ admitting a semi-symmetric non-metric $\phi$-connection whose curvature tensor vanishes, then the scalar curvature $r$ is given by*

$$r = (n-1). \tag{23}$$

**Proof.** In view of equation (22), we have

$$\begin{aligned} {}'\widetilde{R}(X,Y,Z,W) ={}&'R(X,Y,Z,W) + \eta(Z)\eta(Y)g(X,W) - \eta(Z)\eta(X)g(Y,W) \\ &- \eta(Y)\eta(W)g(X,Z) + \eta(X)\eta(W)g(Y,Z) \\ &+ F(X,Z)F(Y,W) - F(Y,Z)F(X,W), \end{aligned} \tag{24}$$

contracting above equation, we have

$$\widetilde{R}ic(Y,Z) = Ric(Y,Z) + (n-1)\eta(Y)\eta(Z). \tag{25}$$

Now since $\widetilde{R}(X,Y,Z) = 0$, so equation (25) becomes

$$Ric(Y,Z) = -(n-1)\eta(Y)\eta(Z), \tag{26}$$

which gives

$$r = (n-1).$$

Thus we have the statement.

**Theorem 4.2** *An LP-Sasakian manifold admitting a semi-symmetric non-metric $\phi$-connection $\widetilde{B}$ is closed with respect to semi-symmetric non-metric $\phi$-connection $\widetilde{B}$ if and only if it is closed with respect to Riemannian connection $D$.*

**Proof.** We have

$$\begin{aligned} X(F(Y,Z)) &= (\widetilde{B}_X F)(Y,Z) + F(\widetilde{B}_X Y, Z) + F(Y, \widetilde{B}_X Z) \\ &= (D_X F)(Y,Z) + F(D_X Y, Z) + F(Y, D_X Z), \end{aligned} \tag{27}$$

using (15) in above equation, we obtain

$$\begin{aligned} (\widetilde{B}_X F)(Y,Z) ={}&(D_X F)(Y,Z) - \eta(Y)F(\phi X, Z) \\ &- \eta(Z)F(\phi X, Y) + 2\eta(X)F(\phi Y, Z). \end{aligned} \tag{28}$$

From (28), we have

$$(\widetilde{B}_X F)(Y,Z) + (\widetilde{B}_Y F)(Z,X) + (\widetilde{B}_Z F)(X,Y) = (D_X F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(Z,X), \tag{29}$$

which proves the statement.

**Theorem 4.3** *An LP-Sasakian manifold admitting a semi-symmetric non-metric $\phi$-connection is such that $\widetilde{B}_X F = 0$,then $F$ is killing if and only if*

$$\eta(X)g(Y,Z) + \eta(Y)g(X,Z) = 2\eta(Z)g(X,Y). \tag{30}$$

**Proof.** Since $\widetilde{B}_X F = 0$, so in view of (28) we have

$$(D_X F)(Y,Z) = -2\eta(X)F(\phi Y,Z) + \eta(Y)F(\phi X,Z) + \eta(Z)F(\phi X,Y). \tag{31}$$

Similarly,

$$(D_Y F)(X,Z) = -2\eta(Y)F(\phi X,Z) + \eta(X)F(\phi Y,Z) + \eta(Z)F(\phi Y,X). \tag{32}$$

Adding (31) and (32)and then using (3), we get

$$(D_X F)(Y,Z) + (D_Y F)(X,Z) = -\eta(X)g(Y,Z) - \eta(Y)g(X,Z) + 2\eta(Z)g(X,Y), \tag{33}$$

which proves the statement.

We know that conformal curvature tensor $V$, con-harmonic curvature tensor $L$ and con-circular curvature tensor $C$ are defined as follows

$$V(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)RY + g(Y,Z)RX\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{34}$$

$$L(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}\{Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)RY + g(Y,Z)RX\}, \tag{35}$$

$$C(X,Y,Z) = R(X,Y,Z) - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{36}$$

**Theorem 4.4** *An LP-Sasakian manifold admitting a semi-symmetric non-metric $\phi$-connection whose curvature tensor is of the form*

$$\begin{aligned}\widetilde{R}(X,Y,Z) =& 2F(X,Y)\phi Z + F(X,Z)\phi Y - F(Y,Z)\phi X \\ & - 2g(X,Z)\eta(Y)\xi - 2\eta(X)\eta(Z)Y \\ & + 2g(Y,Z)\eta(X)\xi + 2\eta(Y)\eta(Z)X,\end{aligned} \tag{37}$$

*then conformal curvature tensor $V$ concides with the con-harmonic curvature tensor $L$ and con-circular curvature tensor $C$ concides with Riemannian curvature tensor $R$.*

**Proof.** Since $\widetilde{R}(X,Y,Z)$ is of the form (37), so in view of equation (22), we have

$$\begin{aligned}R(X,Y,Z) =& 2g(\phi Y,X)\phi Z - g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi \\ & - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X,\end{aligned} \tag{38}$$

contracting above equation, we get

$$Ric(Y,Z) = n.\eta(Y)\eta(Z) + g(Y,Z), \tag{39}$$

and

$$r = 0. \tag{40}$$

Using (40) in equations (34), (35) and (36) we have the result.

**Theorem 4.5** *If an LP-Sasakian manifold admitting a semi-symmetric non-metric $\phi$-connection whose curvature tensor is of the form*

$$\widetilde{R}(X,Y,Z) = F(X,Z)\phi Y - F(Y,Z)\phi X, \tag{41}$$

*then $M_n$ is conformally flat.*

**Proof.** Since $\widetilde{R}(X,Y,Z)$ is of the form (41), so in view of equation (22) we have

$$\begin{aligned}R(X,Y,Z) =& -\eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y \\ & - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi,\end{aligned} \tag{42}$$

contracting equation (42), we get

$$Ric(Y, Z) = g(Y, Z) - (n - 2)\eta(Y)\eta(Z), \tag{43}$$

and

$$r = 2(n - 1). \tag{44}$$

Using (42), (43) and (44) in equation (34), we have the statement.

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