Second Order \((b,F)\)-Convexity in Multiobjective Nonlinear Programming

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Abstract
A new class of functions namely, second order \((b,F)\)-convex functions, which is an extension of \((b,F)\)-convex functions and second order \(F\)-convex functions, is introduced. Sufficient optimality conditions for proper efficiency and second order mixed type duality theorems for multiobjective nonlinear programming problems are established under the assumptions of second order \((b,F)\)-convexity on the functions involved.

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1 Introduction

The importance of convex functions is well known in optimization theory. But for many mathematical models used in decision sciences, applied mathematics and engineering, the notion of convexity does no longer suffice. So it is possible to generalize the notion of convexity and to extend the validity of results to larger class of optimization problems. Consequently, various generalizations of convex functions have been introduced in the literature. The field of multiobjective programming, also known as vector programming, has grown remarkably in different directions in the setting of optimality conditions and duality theory since the 1980’s. It has been enriched by the applications of various types of
generalizations of convexity theory, with and without differentiability assumptions, and in the framework of continuous time programming, fractional programming, inverse vector optimization, saddle point theory, symmetric duality, variational problems etc.

Many researchers [1, 2, 5-8, 11-13, 15, 19, 22, 24-26] have used proper efficiency and efficiency to establish optimality conditions and duality results for multiobjective programming problems under different assumptions of convexity. A second order dual for a nonlinear programming problem was introduced by Mangasarian [18] and established duality results for nonlinear programming problems. Mond [14] introduced the concept of second order convex functions and proved second order duality under the assumptions of second order convexity on the functions involved. Mond and Zhang [16] established various duality results for multiobjective programming problems involving second order V-invex functions. Zhang and Mond [27] introduced second order F-convex functions as a generalization of F-convex functions [10] and obtained various second order duality results for multiobjective nonlinear programming problems under the assumption of second order F-convexity. Suneja et al. [23] obtained duality results for multiobjective programming under the assumption of \( \eta \)-bonvexity and its related functions. Ahmed [3] obtained optimality conditions and mixed duality results for nondifferentiable programming problems. Ahmed and Husain [4] obtained second order Mond-Weir type dual results for multiobjective programming problems under the assumption of second order \( (\alpha,\rho,d) \)-convexity on the functions involved. The study of the second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective functions when approximations are used [14,18, 9].

In this paper, we introduce a new class of functions namely, second order \((b,F)\)-convex functions which is an extension of \((b,F)\)-convex functions [20] and second order F-convex functions [27]. Then, we derive sufficient optimality conditions for proper efficiency and obtain second order mixed type duality theorems for multiobjective nonlinear programming problems under the assumption of second order \((b,F)\)-convexity.

2 Preliminaries

Throughout this paper, the following conventions for vectors in \( \mathbb{R}^n \) will be followed. For any \( x = (x_1, x_2, \ldots, x_n)^T \) and \( y = (y_1, y_2, \ldots, y_n)^T \), we follow the notations of Mangasarian [17]:

\[
x = y \quad \text{if and only if} \quad x_i = y_i, \quad i = 1, 2, \ldots, n; \quad x < y \quad \text{if and only if} \quad x_i < y_i, \quad i = 1, 2, \ldots, n; \quad x \leq y \quad \text{if and only if} \quad x_i \leq y_i, \quad i = 1, 2, \ldots, n; \quad x < y \quad \text{if and only if} \quad x_i < y_i, \quad i = 1, 2, \ldots, n \quad \text{and} \quad x < y_r \quad \text{for some} \quad r \in \{1, 2, \ldots, n\} \quad \text{and} \quad x \not\leq y \quad \text{is the negation of} \quad x \leq y.
\]
Let $X$ be an open convex subset of $\mathbb{R}^n$ and $R_+$ denote the set of all positive real numbers and $e = (1,1,...,1) \in \mathbb{R}^k$. Let us assume that $h : X \rightarrow R$, $f : X \rightarrow \mathbb{R}^k$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ where $f = (f_1,...,f_k)$ and $g = (g_1,...,g_m)$ are twice differentiable functions on $X$ and $f_i$ and $g_j : X \rightarrow \mathbb{R}$, for all $i = 1,2,...,k$ and $j = 1,2,...,m$. Let the vectors $p = (p_1,p_2,...,p_n)^T \in \mathbb{R}^n$, $\lambda = (\lambda_1,\lambda_2,...,\lambda_k) \in \mathbb{R}^k$ and $y = (y_1,y_2,...,y_m) \in \mathbb{R}^m$. Let $F$ be a function defined by $F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$ and the functions $b_i(x,u)$ and $c_j(x,u) : X \times X \rightarrow R_+$ and $h_i(x,u)$ and $c_j(x,u) : X \times X \rightarrow R_+$, for all $i = 1,2,...,k$ and $j = 1,2,...,m$.

Consider the following multiobjective nonlinear programming problem

(MOP) \text{Minimize } f(x) = (f_1(x),f_2(x),...,f_k(x))

subject to \text{ } g(x) \leq 0, \text{ } x \in X,

where $f_i : X \rightarrow R, i = 1,2,...,k$ and $g : X \rightarrow \mathbb{R}^m$ where $g = (g_1,...,g_m)$ are twice differentiable functions on $X$.

Let $P = \{x \in X : g_j(x) \leq 0, j = 1,2,...,m\}$. That is, $P$ is the set of all feasible solutions for the problem (MOP).

We need the following definitions which can be found in [10, 5, 6, 8].

**Definition 1:** A function $F : X \times X \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be sublinear in its third argument if for each $x,u \in X$,

$F(x,u;a + b) \leq F(x,u;a) + F(x,u;b)$, for all $a,b \in \mathbb{R}^n$ and

$F(x,u;\alpha a) = \alpha F(x,u;a)$, for all $\alpha \geq 0$ in $R$ and $a \in \mathbb{R}^n$.

**Note:** $F(x,u;0) = 0$, for all $x,u \in X$.

**Definition 2:** A feasible point $x^o$ is said to be efficient for (MOP) if there exists no other feasible point $x$ in (MOP) such that $f_i(x) \leq f_i(x^o)$, $i = 1,2,...,k$ and $f_i(x) < f_i(x^o)$, for some $r \in \{1,2,...,k\}$.

**Definition 3:** A feasible point $x^o$ is said to be a properly efficient solution of (MOP), if it is efficient and if there exists a scalar $M > 0$ such that, for each $i \in \{1,2,...,k\}$ and for all feasible $x$ of (MOP) satisfying $f_i(x) < f_i(x^o)$, we have $f_i(x^o) - f_i(x) \leq M(f_i(x) - f_i(x^o))$ for some $r$ such that $f_i(x) > f_i(x^o)$.

We need the following theorem for proving sufficient optimality conditions for proper efficiency and duality theorems which can be found in [12].
Theorem 1: Let $\lambda^* > 0$ in $\mathbb{R}^k$ be fixed with $\lambda^T e = 1$. If $x^*$ is an optimal solution of the scalar programming problem $(MOP_{\lambda^*})$ where

$$(MOP_{\lambda^*}) \text{ Minimize } \lambda^T f(x), \ x \in P,$$

then $x^*$ is a properly efficient solution for $(MOP)$.

We need the following necessary optimality conditions for proving strong duality theorem, which can be found in Pandian [21].

Theorem 2: (Necessary Optimality Conditions): Assume that $x^*$ is an efficient solution for $(MOP)$ at which a constraint qualification [17] is satisfied for each $(MOP_{r}(x^*))$, $r \in \{1,2,\ldots,k\}$ where

$$(MOP_{r}(x^*)) \text{ Minimize } f_i(x)$$

subject to $f_i(x) \leq f_i(x^*)$, for all $i \neq r, \ x \in P$.

Then, there exist scalars $\lambda^* > 0$ in $\mathbb{R}^k$ with $\lambda^T e = 1$ and $y^* \geq 0$ in $\mathbb{R}^m$ such that $(x^*, \lambda^*, y^*)$ satisfies

$$\sum_{i=1}^{k} \lambda_i^* \nabla f_i(x^*) + \sum_{j=1}^{m} y_j^* \nabla g_j(x^*) = 0 \quad (1)$$

$$y_j^* g_j(x^*) = 0, \ j = 1,2,\ldots,m. \quad (2)$$

3 Second order $(b,F)$-convex functions

We, now, define a new class of functions namely, second order $(b,F)$-convex functions, which is a generalization of $(b,F)$-convex functions [20] and second order $F$-convex functions [27].

Definition 4: The function $h$ is said to be second order $(b_F,F)$-convex at $u \in X$ with respect to $b_u(x,u)$ if for all $x \in X$ and $p \in \mathbb{R}^n$,

$$b_u(x,u)[h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u)p] \geq F(x,u; \nabla h(u) + \nabla^2 h(u)p).$$

Remark 1: If $p = 0$, then the above definition reduces to the definition of $(b,F)$-convex function [20].

Remark 2: Every second order $F$-convex function is a second order $(b,F)$-convex function with $b_u(x,u)=1$, but the converse is not true. This is demonstrated in Example 1.
Remark 3: Every second order convex function is a second order \((b,F)\)-convex function with \(b(x,u)=1\) and \(F(x,u;z)=(x-u)^T z\), but the converse is not true. This is demonstrated in Example 1.

Example 1: Let \(X = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 - 5 < 0\}\) and \(p \in \mathbb{R}^2\). Define \(h : X \to \mathbb{R}, \quad F : X \times X \times \mathbb{R}^2 \to \mathbb{R}\) and \(b_\circ(x,u) : X \times X \to \mathbb{R}_+\) as follows,

\[
h(x) = \sin x_1 + \sin x_2, \quad F(x,u;z) = \left(\|z_1\| + \|z_2\|\right)(h(x) - h(u)) \quad \text{and} \quad b_\circ(x,u) = 2; u_1 = u_2 \quad \text{and} \quad b_\circ(x,u) = 0; u_1 \neq u_2,
\]

where \(x = (x_1,x_2), \ u = (u_1,u_2)\) and \(z = (z_1,z_2)\).

Now, for \(u = (0,0), \) we have for all \(x \in X\) and \(p \in \mathbb{R}^2\), \(b_\circ(x,u)[h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u)p] \geq F(x,u;\nabla h(u) + \nabla^2 h(u)p).\)

Therefore, \(h\) is second order \((b_\circ,F)\)-convex at \(u = (0,0)\) with respect to \(b_\circ(x,u).\)

Now, for \(u = (0,0), \ x = (1,1)\) and any \(p \in \mathbb{R}^2\), we have \(h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u)p < F(x,u;\nabla h(u) + \nabla^2 h(u)p).\)

Therefore, \(h\) is not second order \(F\)-convex at \(u = (0,0)\).

Now, for \(u = (0,0), \ x = (1,0)\) and any \(p \in \mathbb{R}^2\), we have \(h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u)p < (x-u)^T \nabla h(u) + \nabla^2 h(u)p.\)

Therefore, \(h\) is not second order convex at \(u = (0,0)\).

Definition 5: The function \(h\) is said to be strictly second order \((b_\circ,F)\)-convex at \(u \in X\) with respect to \(b_\circ(x,u)\) if for all \(x \in X, \ x \neq u\) and \(p \in \mathbb{R}^n, \)

\[
b_\circ(x,u)[h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u)p] > F(x,u;\nabla h(u) + \nabla^2 h(u)p).\]

4 Sufficient optimality conditions

We, now, prove the sufficient optimality conditions for a feasible point of the problem (MOP) to be a properly efficient solution under the assumption of second order \((b,F)\)-convexity on the function involved.

Theorem 3: (Sufficient Optimality Conditions) Let \(x^*\) be feasible for (MOP) and there exists scalars \(\lambda^* > 0\) in \(\mathbb{R}^k\) with \(\lambda^* e = 1, \ y_j^* \geq 0\) in \(\mathbb{R}, \ j \in I(x^*)\) and \(p^* \in \mathbb{R}^n\) such that

\[
\sum_{i=1}^k \lambda_i \nabla f_i(x^*) + \sum_{j \in I(x^*)} y_j^* \nabla g_j(x^*) + \left[ \sum_{i=1}^k \lambda_i \nabla^2 f_i(x^*) \right] \sum_{j \in I(x^*)} y_j^* \nabla^2 g_j(x^*) \right] p^* = 0, \quad (3)
\]
where \( I(x^\circ) = \{j : g_j(x^\circ) = 0\} \). If \( \lambda^T f \) is second order \((b_c,F)\)-convex at \( x^\circ \) with respect to \( b_c(x,x^\circ) \) with \( b_c(x,x^\circ) > 0 \) and each \( g_j, j \in I(x^\circ) \) is second order \((c_j,F)\)-convex at \( x^\circ \) with respect to \( c_j(x,x^\circ) \), then \( x^\circ \) is a properly efficient solution for (MOP) provided that \( p^T (\nabla^2 \lambda^T f(x^\circ)) p^\circ \leq 0 \) and \( p^T (\nabla^2 g_j(x^\circ)) p^\circ \leq 0 \), for all \( j \in I(x^\circ) \).

**Proof.** Let \( x \) be a feasible solution of (MOP).

Now, since \( x^\circ \) is feasible for (MOP) and \( y_j^\circ \geq 0 \) and \( p^T (\nabla^2 g_j(x^\circ)) p^\circ \leq 0 \), for all \( j \in I(x^\circ) \) and by the second order \((c_j,F)\)-convexity of \( g_j \) at \( x^\circ \), for all \( j \in I(x^\circ) \) and since \( y_j^\circ \geq 0 \), \( j \in I(x^\circ) \) and \( F \) is sublinear, we have

\[
F(x,x^\circ; \nabla( \sum_{j \in I(x^\circ)} y_j^\circ g_j(x^\circ) ) + \nabla^2( \sum_{j \in I(x^\circ)} y_j^\circ g_j(x^\circ) ) p^\circ ) \leq 0. \tag{4}
\]

Now, by the sublinearity of \( F \) and from (3), we have

\[
F(x,x^\circ; \sum_{i=1}^k \lambda_i \nabla f_i(x^\circ) + (\sum_{i=1}^k \lambda_i \nabla^2 f_i(x^\circ)) p^\circ ) \geq -F(x,x^\circ; \sum_{j \in I(x^\circ)} y_j^\circ \nabla g_j(x^\circ) + (\sum_{j \in I(x^\circ)} y_j^\circ \nabla^2 g_j(x^\circ) ) p^\circ ). \tag{5}
\]

Now, from (4) and (5), by the second order \((b_c,F)\)-convexity of \( \lambda^T f \) and since \( p^T (\nabla^2 \lambda^T f(x^\circ)) p^\circ \leq 0 \) and \( b_c(x,x^\circ) > 0 \), we can conclude that

\( \lambda^T f(x) \geq \lambda^T f(x^\circ) \).

Thus, \( x^\circ \) is an optimal solution of \((MOP)_\lambda\). By the Theorem 1, \( x^\circ \) is a properly efficient solution for (MOP).

Hence the Theorem.

5. Mixed Duality theorems

Let \( J_1 \) be a subset of \( M = \{1,2,...,m\} \) and \( J_2 = M \setminus J_1 \). We consider the following second order mixed type dual for (MOP).

\textbf{(XMOP)} Maximize \( f(u) + y_{j_1} g_{j_1}(u) e - \frac{1}{2} p^T \nabla^2 [f(u) + y_{j_1} g_{j_1}(u)] p \)

subject to

\[
\nabla \lambda^T f(u) + (\nabla^2 \lambda^T f(u)) p + \nabla y^T g(u) + (\nabla^2 y^T g(u)) p = 0
\]

\[
y_{j_2} g_{j_2}(u) - \frac{1}{2} p^T (\nabla^2 y_{j_2} g_{j_2}(u)) p \geq 0
\]

\[
y \geq 0, \lambda > 0, \lambda^T e = 1,
\]
where \( y_{J_1} g_{J_1} = \sum_{j=J_1} y_j g_j(u) \) and \( y_{J_2} g_{J_2} = \sum_{j=J_2} y_j g_j(u) \).

Note: We get a second order Mond-Weir type dual [27] for \( J_1 = \phi \) and a second order Mangasarian type dual [17,18] for \( J_2 = \phi \) in (XMOP) respectively.

We, now, prove weak duality theorem and strong duality theorem between the problems (MOP) and (XMOP) under the assumptions of second order \((b, F)\)-convexity on the functions involved.

**Theorem 4: (Weak Duality Theorem)** Let \( x \) be feasible for (MOP) and \((u, \lambda, y, p)\) be feasible for (XMOP). If \( \lambda^T f + y_{J_1} g_{J_1} \) is second order \((b, F)\)-convex at \( u \) with respect to \( b(x,u) \) with \( b(x,u) > 0 \) and \( y_{J_2} g_{J_2} \) is second order \((c, F)\)-convex at \( u \) with respect to \( c(x,u) \), then

\[
\lambda^T f(x) \geq \lambda^T f(u) + y_{J_1} g_{J_1}(u) - \frac{1}{2} p^T \nabla^2 [\lambda^T f(u) + y_{J_1} g_{J_1}(u)] p.
\]

**Proof.** Now, since \( \lambda^T f + y_{J_1} g_{J_1} \) is second order \((b, F)\)-convex at \( u \), we have

\[ b(x,u)[\lambda^T f(x) + y_{J_1} g_{J_1}(x) - \lambda^T f(u) - y_{J_1} g_{J_1}(u)] + \frac{1}{2} p^T \nabla^2 (\lambda^T f(u) + y_{J_1} g_{J_1}(u)) p \]

\[
\geq F(x,u; \nabla (\lambda^T f(u) + y_{J_1} g_{J_1}(u)) + \nabla^2 (\lambda^T f(u) + y_{J_1} g_{J_1}(u)) p). \tag{6}
\]

Now, since \( x \) is feasible for (MOP) and \((u, \lambda, y, p)\) is feasible for (XMOP), and since \( y_{J_2} g_{J_2} \) is second order \((c, F)\)-convex at \( u \) with respect to \( c(x,u) \), we have

\[ F(x,u; \nabla (y_{J_2} g_{J_2}(u)) + \nabla^2 (y_{J_2} g_{J_2}(u)) p) \leq 0. \tag{7}
\]

Now, from (7) and since \((u, \lambda, y, p)\) is feasible for (XMOP) and \( F \) is sublinear, we have

\[ F(x,u; \nabla (\lambda^T f(u) + y_{J_1} g_{J_1}(u)) + \nabla^2 (\lambda^T f(u) + y_{J_1} g_{J_1}(u)) p) \geq 0. \tag{8}
\]

Now, from (6) and (8) and since \( b(x,u) > 0 \) and \( y_{J_1} g_{J_1}(x) \leq 0 \), we have

\[ \lambda^T f(x) \geq \lambda^T f(u) + y_{J_1} g_{J_1}(u) - \frac{1}{2} p^T \nabla^2 [\lambda^T f(u) + y_{J_1} g_{J_1}(u)] p. \]

Hence the theorem.

**Theorem 5: (Strong Duality Theorem)** Assume that \( x^* \) is an efficient solution for (MOP) at which a constraint qualification [17] is satisfied for each \((\text{MOP}, (x^*))\), \( r \in \{1, 2, \ldots, k\} \). Then, there exists \( \lambda^* \in \mathbb{R}^k \), \( y^* \in \mathbb{R}^m \) such that \((x^*, \lambda^*, y^*, p^* = 0)\) is a feasible solution for (XMOP) and the corresponding objective function values of (MOP) and ( XMOP) are equal. If the conditions of the Theorem 4. holds, then \((x^*, \lambda^*, y^*, p^* = 0)\) is a properly efficient solution for (XMOP).
Proof. By the Theorem 2, there exist scalars $\lambda > 0$ in $R^k$ with $\lambda^Te = 1$ and $y^* \geq 0$ in $R^m$ such that $(x^*, \lambda^*, y^*, p^*)$ satisfies (1) and (2). Therefore, $(x^*, \lambda^*, y^*, p^*)$ is feasible for (XMOP) and the objective value of the problem (MOP) at $x^*$ and the objective value of (XMOP) at $(x^*, \lambda^*, y^*, p^*)$ are equal.

Suppose that $(x^*, \lambda^*, y^*, p^*)$ is not efficient for (XMOP). Then, there exists a feasible $(u, \lambda, y, p)$ for (XMOP) such that

$$f(x^*) \leq f(u) + y_{j_1}^T g_{j_1}(u)e - \frac{1}{2} p^T \nabla^2 [f(u) + y_{j_1}^T g_{j_1}(u)]p.$$  

Since $\lambda > 0$ in $R^k$ with $\lambda^Te = 1$, it follows that

$$\lambda^T f(x^*) < \lambda^T f(u) + y_{j_1}^T g_{j_1}(u)e - \frac{1}{2} p^T \nabla^2 [\lambda^T f(u) + y_{j_1}^T g_{j_1}(u)]p,$$

which contradicts the Theorem 4. Thus, $(x^*, \lambda^*, y^*, p^*)$ is an efficient solution for (XMOP).

Suppose that $(x^*, \lambda^*, y^*, p^*)$ is not properly efficient for (XMOP). Then, for every $M > 0$, there exists a feasible solution $(u, \lambda, y, p)$ of (XMOP) and an index $i$ such that $f_i(u) + y_{j_1}^T g_{j_1}(u) - f_i(x^*) > M[f_i(x^*) - f_i(u) - y_{j_1}^T g_{j_1}(u)]$ for all $r$ satisfying $f_i(x^*) - f_i(u) - y_{j_1}^T g_{j_1}(u) > 0$ whenever $f_i(u) + y_{j_1}^T g_{j_1}(u) - f_i(x^*) > 0$. This means that $f_i(u) + y_{j_1}^T g_{j_1}(u) - f_i(x^*)$ can be made arbitrarily large. Since $\lambda > 0$ in $R^k$ with $\lambda^Te = 1$, $\lambda^T f + y_{j_1}^T g_{j_1}$ is second order $(b,F)$-convex at $u$ with respect to $b_i(x,u)$ with $b_i(x,u) > 0$. Since $(u,\lambda,y,p)$ is feasible for (XMOP) and $F$ is sublinear, we can conclude that

$$F(x^*, u; \nabla(y_{j_2}^T g_{j_2}(u))) + \nabla^2 (y_{j_2}^T g_{j_2}(u))p > 0.$$  

Now, since $x^*$ is feasible for (MOP) and $(u,\lambda,y,p)$ is feasible for (XMOP) and by the second order $(c,F)$-convexity of $y_{j_2}^T g_{j_2}$ at $u$, we have

$$F(x^*, u; \nabla(y_{j_2}^T g_{j_2}(u))) + \nabla^2 (y_{j_2}^T g_{j_2}(u))p \leq 0,$$

which contradicts (9). Thus, $(x^*, \lambda^*, y^*, p^*)$ is a properly efficient solution for (XMOP).

Hence the theorem.

6. Strict converse mixed duality theorems

We now prove the following strict converse duality theorems between the problems (MOP) and (XMOP) under the assumption of second order $(b,F)$-convexity on the functions involved.
Theorem 6: (Strict Converse Duality Theorem) Let $x^*$ be feasible for (MOP) and $(u^*, \lambda^*, y^*, p^*)$ be feasible for (XMOP) such that

$$
\lambda^T f(x^*) = \lambda^T f(u^*) + y^*_1 g_{J1}(u^*) - \frac{1}{2} p^T \nabla^2 (\lambda^T f(u^*) + y^*_1 g_{J1}(u^*))p^*.
$$

(10)

If $\lambda^T f + y^*_1 g_{J1}$ is strictly second order $(b_o, F)$-convex at $u^*$ with respect to $b_o(x^*, u^*)$ with $b_o(x^*, u^*) > 0$ and $y^*_2 g_{J2}$ is second order $(c_o, F)$-convex at $u^*$ with respect to $c_o(x^*, u^*)$, then $x^* = u^*$ and $u^*$ is a properly efficient solution for (MOP).

Proof. Suppose that $x^* \neq u^*$.

Now, since $x^*$ is feasible for (MOP), $(u^*, \lambda^*, y^*, p^*)$ is feasible for (XMOP) and $y^*_2 g_{J2}$ is second order $(c_o, F)$-convex at $u^*$ and also, since $(u^*, \lambda^*, y^*, p^*)$ is feasible for (XMOP) and $F$ is sublinear, we have

$$
F(x^*, u^*; \nabla (\lambda^T f(u^*) + y^*_1 g_{J1}(u^*))) + \nabla^2 (\lambda^T f(u^*) + y^*_1 g_{J1}(u^*))p^* \geq 0.
$$

Since $\lambda^T f + y^*_1 g_{J1}$ is strictly second order $(b_o, F)$-convex at $u^*$ with $b_o(x^*, u^*) > 0$ and $y^*_1 g_{J1}(x^*) \leq 0$, we can conclude that

$$
\lambda^T f(x^*) > \lambda^T f(u^*) + y^*_1 g_{J1}(u^*) - \frac{1}{2} p^T \nabla^2 (\lambda^T f(u^*) + y^*_1 g_{J1}(u^*))p^*,
$$

which contradicts (10). Thus, $x^* = u^*$.

Now, since $x^* = u^*$ and from the Theorem 4., Theorem 1. and (10), we can conclude that $u^*$ is a properly efficient solution for (MOP).

Hence the theorem.

Theorem 7: (Strict Converse Duality Theorem) Let $x^*$ be feasible for (MOP) and $(u^*, \lambda^*, y^*, p^*)$ be feasible for (XMOP) such that (10) is satisfied. If $\lambda^T f + y^*_1 g_{J1}$ is second order $(b_o, F)$-convex at $u^*$ with respect to $b_o(x^*, u^*)$ and $y^*_2 g_{J2}$ is strictly second order $(c_o, F)$-convex at $u^*$ with respect to $c_o(x^*, u^*)$, then $x^* = u^*$ and $u^*$ is a properly efficient solution for (MOP).

Proof. Suppose that $x^* \neq u^*$.

Now, since $x^*$ is feasible for (MOP) and $(u^*, \lambda^*, y^*, p^*)$ is feasible for (XMOP) and also, since $y^*_2 g_{J2}$ is strictly second order $(c_o, F)$-convex at $u^*$, it follows that

$$
F(x^*, u^*; \nabla (y^*_1 g_{J2}(x^*))) + \nabla^2 (y^*_1 g_{J2}(u^*))p^* < 0.
$$

(11)
Now, since \((u^0, \lambda^0, y^0, p^0)\) is feasible for (XMOP) and \(F\) is sublinear, we have
\[
F(x^0, u^0; \nabla (\lambda^T f(u^0) + y^T g(u^0)) + \nabla^2 (\lambda^T f(u^0) + y^T g(u^0)) p^0) = 0.
\] (12)

Now, from (10) and since \(y^0_j g_{j_1}(x^0) \leq 0\) and \(\lambda^T f + y^0_j g_{j_1}\) is second order \((b, F)\)-convex at \(u^0\), we can conclude that
\[
F(x^0, u^0; \nabla (\lambda^T f(u^0) + y^0_j g_{j_1}(u^0)) + \nabla^2 (\lambda^T f(u^0) + y^0_j g_{j_1}(u^0)) p^0) \leq 0.
\] (13)

Now, from (11) and (13) and since \(F\) is sublinear, we have,
\[
F(x^0, u^0; \nabla (\lambda^T f(u^0) + y^T g(u^0)) + \nabla^2 (\lambda^T f(u^0) + y^T g(u^0)) p^0) < 0,
\]
which contradicts (12). Thus, \(x^0 = u^0\).

Now, since \(x^0 = u^0\) and from the Theorem 4., the Theorem 1. and (10), we can conclude that \(u^0\) is a properly efficient solution for (MOP).

Hence the theorem.

References


Second order \((b,F)\)-convexity


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