The General Eulerian Integral

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Abstract

In the process of study of the screening properties of a charged impurity located near and inside the surface of the metal subjected to a magnetic field, an interesting and useful class of Eulerian integral including the Bessel function $J_v(z)$ or $J_0(z)$ arose, which were written in compact forms by M.L. Glasser and further generalized and extended by L.T. Wille as giving compact form expressions for a number of Eulerian integrals pertaining to Meijer’s G-function. H.M. Srivastava (1993) gave the generalization of these results. Getting inspired by this work we aim to evaluating the general class of Eulerian integrals involving I-function and general class of polynomials. Our main result (2.4) below is shown to provide key formula from which many integrals can be deduced.

Mathematics Subject Classification: 33C60, 33C45

Keywords: Beta Integral, I-function, general class of polynomials.
1 Introduction and Preliminaries

In the theory of Gamma and Beta functions, it is well known that the Eulerian beta integral

\[ B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} \, dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad [\text{Re}(\alpha) > 0; \text{Re}(\beta) > 0] \quad (1.1) \]

can be rewritten in its equivalent form

\[ \int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} \, dt = (b-a)^{\alpha+\beta-1} B(\alpha, \beta) \quad [\text{Re}(\alpha) > 0; \text{Re}(\beta) > 0; a \neq b] \quad (1.2) \]

since

\[ (ut+v)^\gamma = (au+v)^\gamma \sum_{l=0}^{\infty} \frac{(-\gamma)_l}{l!} \left\{ -\frac{(t-a)u}{au+v} \right\}^l \quad [|(t-a)u| < |au+v|; t \in [a,b]] \quad (1.3) \]

We readily find from (1.2) that [2, p. 301, eq. 2.2.6.1]

\[ \int_a^b (t-a)^{\alpha-1} (b-t)^{\beta-1} (ut+v)^\gamma \, dt \]

\[ = (b-a)^{\alpha+\beta-1} (au+v)^\gamma B(\alpha, \beta) \binom{2}{\alpha, -\gamma; \alpha + \beta; -\frac{(b-a)u}{au+v}} \quad (1.4) \]

\[ \left[ \text{Re}(\alpha) > 0; \text{Re}(\beta) > 0; \left| \arg \left( \frac{bu+v}{au+v} \right) \right| \leq \pi - \varepsilon (0 < \varepsilon < \pi); a \neq b \right] \]

For \( \gamma = -\alpha - \beta \), the second member of (1.4) would simplify considerably, and if further we set \( u = \lambda - \mu \) and \( v = (1 + \mu) b - (1 + \lambda) a \) in terms of new parameters \( \lambda \) and \( \mu \), the special case \( \gamma = -\alpha - \beta \) of (1.4) would yield (Gradshteyn, 1980[8]p.287,eq.3.198); see also (Prudnikov et al, 1983 [2, p.301, eq.2.2.6.1])

\[ \int_a^b \frac{(t-a)^{\alpha-1} (b-t)^{\beta-1}}{b-a + \lambda (t-a) + \mu (b-t)} \lambda^{\lambda+\mu} \, dt = \frac{(1+\lambda)^{-\alpha} (1+\mu)^{-\beta}}{(b-a)} B(\alpha, \beta) \quad (1.5) \]

\[ [\text{Re}(\alpha) > 0; \text{Re}(\beta) > 0; b-a + \lambda (t-a) + \mu (b-t) \neq 0, t \in [a,b]; a \neq b] \]

Making use of (1.5) we address the problem of closed form evaluation of the following general Eulerian integral:

\[ \Re = \int_{I}^{w} \frac{(t-l)^{\lambda} (w-t)^{\mu}}{f(t)} \lambda^{\lambda+\mu+2} \int_{P_{1},Q_{1},r}^{M_{N}} \left[ z \{ g(t) \} \right]^{v} \left[ a_{j}, \alpha_{j} \right]_{1,N} \left( a_{j}, \alpha_{j} \right)_{N+1,P_{i}} \left[ b_{j}, \beta_{j} \right]_{1,M} \left( b_{j}, \beta_{j} \right)_{M+1,Q_{i}} \]
During the course of the study, we shall also require the general class of polynomials \( S^U_V \) defined by Srivastava \[6\]

\[
S^U_V [x] = \sum_{R=0}^{[V/U]} (-V)^{UR} A(V, R) \frac{2^R}{R!}
\]

where \( U \) is an arbitrary positive integer, \( V = 0, 1, 2 \), and the coefficients \( A(V, R) \) are arbitrary constants, real or complex. A number of well known polynomials follow as special
cases of $S^U_V$ as referred in [3].

The generalized polynomial set is defined by the following Rodrigues type formula [10, p.64, eq. (2.18)]

\[ S_n^{\mu,\delta,\tau} [x; w, s, q, A, B, m, \xi, l] = (Ax + B)^{-\mu} (1 - \tau x^w)^{-\delta/\tau} T_{\xi,l}^{m+n} \left( (Ax + B)^{\mu+q} (1 - \tau x^w)^{\delta/\tau + sn} \right) \]  

with the differential operator

\[ T_{k,l} = x^l \left[ k + x \frac{d}{dx} \right] \]

The explicit series form of this generalized sequence of functions is given by [11, p.71, eq. (2.3.4)].

\[ S_n^{\mu,\delta,\tau} [x; w, s, q, A, B, m, \xi, l] = B^{mn} x^{l(m+n)} (1 - \tau x^w)^{sn} \sum_{\sigma=0}^{m+n} \sum_{r=0}^{m+n} \sum_{j=0}^{m+n} \sum_{i=0}^{j} \frac{(-1)^j}{\sigma! r! j! i! (1 - \mu - j)_i} \]

\[ \times \left( -\frac{\delta}{\tau} - sn \right)^\sigma \left( i + \xi + wr \right) \frac{(-\tau x^w)^\sigma}{1 - \tau x^w} \left( Ax \right)^i \]

Some special cases of (1.13) are given by Raijada in table form [11, p.65]. We shall use the following special case:

If we put $A=1, B=0$ in (1.13) and letting $\tau \to 0$ and using the well known results $Lt_{\tau \to 0} (1 - \tau x^w)^{\delta/\tau} = \exp (-\delta x^w), \; Lt_{|b| \to \infty} \left( \frac{z}{b} \right)^n = z^n$. Therein, we arrive at the following important polynomial set

\[ S_n^{\mu,0} [x] = S_n^{\mu,0} [x; w, q, 1, 0, m, \xi, l] \]

\[ = x^{qn+l(m+n)} p_{m+n} \sum_{\sigma=0}^{m+n} \sum_{r=0}^{m+n} (-\sigma) \frac{(-\tau x^w)^\sigma}{1 - \tau x^w} \frac{(Ax)^i}{r! \sigma!} \]  

\[ \left( \frac{u+q n + \xi + \tau e}{l} \right)^{m+n} \left( \beta x^r \right)^n \]

\section{Evaluation of our main integral (1.6)}

Expressing the general Eulerian integral (1.6) making use of definition (1.9), (1.11) and (1.14), we first find from (1.6) that

\[ R = \sum_{R=0}^{\infty} \frac{(-V)_{UR} A_v R}{R!} (y)^R (x)^R h^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{\eta} (-\eta) e \frac{(\alpha'+q n + \xi + \tau e)}{\eta e!} \left( \beta' x^r \right)^n \]
\[ \int_l^{w} \frac{(t-l)^\lambda (w-t)^\mu}{\{f(t)\}^\xi} \left( \frac{1}{2\pi i} \int_L \theta(s) z^s \{g(t)\}^{\nu s+\rho R+\zeta R'+\tau \eta} ds \right) dt \]  

(2.1)

where \( R' = qn + h (m + n) \), \( \Lambda = \rho' R + \zeta R' + \tau \eta \) and \( L \) is a suitable contour of Mellin Barnes type in the complex \( s \) plane and \( f(t), g(t) \) and \( \theta(s) \) are given by (1.7), (1.8) and (1.10) respectively. Assuming the inversion of the order of integration in (2.1) be permissible by absolute (and uniform) convergence of the integral involved above, we have

\[ \Re = \sum_{R=0}^{\infty} \frac{[V]_{URA V, R}}{R!} \left( \frac{y}{\beta^R} \right)^R \left( \frac{x}{\beta^R} \right) \frac{\Gamma(\nu s + \Lambda + k)}{\Gamma(\nu s + \Lambda + k)!} \int_L \frac{1}{\{f(t)\}^{\lambda+\mu+(\gamma+\delta)(\nu s + \Lambda) + 2}} z^s \left\{ \int_l^{w} \frac{(t-l)^\lambda (w-t)^\mu}{\{f(t)\}^\xi} \left( \frac{1}{2\pi i} \int_L \theta(s) z^s \{g(t)\}^{\nu s+\rho R+\zeta R'+\tau \eta} ds \right) dt \right\}^{\nu s-L} ds \]  

(2.2)

If \(|(\beta - \alpha)(t-l)| < |\beta f(t)| \) (\( t \in [l, w] \)) then use can be made of binomial expansion and we thus find from (2.2) that

\[ \Re = \sum_{R=0}^{\infty} \frac{[V]_{URA V, R}}{R!} \left( \frac{y}{\beta^R} \right)^R \left( \frac{x}{\beta^R} \right) \frac{\Gamma(\nu s + \Lambda + k)}{\Gamma(\nu s + \Lambda + k)!} \int_L \frac{1}{\{f(t)\}^{\lambda+\mu+(\gamma+\delta)(\nu s + \Lambda) + k+2}} z^s \left\{ \int_l^{w} \frac{(t-l)^\lambda (w-t)^\mu}{\{f(t)\}^\xi} \left( \frac{1}{2\pi i} \int_L \theta(s) z^s \{g(t)\}^{\nu s+\rho R+\zeta R'+\tau \eta} ds \right) dt \right\}^{\nu s-L} ds \]  

(2.3)

provided also that the order of summation and integration can be inverted. The inner most integral in (2.3) can be evaluated by appealing to the known extension of the Eulerian (beta- function) integral (1.5) and we finally obtain the desired integral formula:

\[ \Re = (w-l)^{-1}(1+\rho)^{-\lambda-\gamma\Lambda-1}(1+\sigma)^{-\mu-\delta\Lambda-1} \sum_{R=0}^{\infty} \frac{[V]_{URA V, R}}{R!} \left( \frac{y}{\beta^R} \right)^R \left( \frac{x}{\beta^R} \right) \frac{\Gamma(\nu s + \Lambda + k)}{\Gamma(\nu s + \Lambda + k)!} \int_L \frac{1}{\{f(t)\}^{\lambda+\mu+(\gamma+\delta)(\nu s + \Lambda) + k+2}} z^s \left\{ \int_l^{w} \frac{(t-l)^\lambda (w-t)^\mu}{\{f(t)\}^\xi} \left( \frac{1}{2\pi i} \int_L \theta(s) z^s \{g(t)\}^{\nu s+\rho R+\zeta R'+\tau \eta} ds \right) dt \right\}^{\nu s-L} ds \]  

\[ \times \prod_{i=1}^{M, N+3, Q_i+2} \left[ \frac{(1+\rho)}{(1+\sigma)^{\delta}} \right]^{-v} (1-k-\Lambda, v) (-k-\lambda-\gamma\Lambda, \gamma v) (-\mu-\delta\Lambda, \delta v) 
\]  

(2.4)
which hold true when

(i) \( V > 0; \gamma > 0; \delta > 0; \beta \neq 0; w \neq l; \rho, \sigma \neq -1 \) and \( \{ w - l + \rho (t - l) + \sigma (w - t) \} \neq 0, t \in [l, w] \).

(ii) \( \text{Re} \left\{ 1 + \lambda + \gamma v \left( \frac{b_j}{\beta_j} \right) \right\} > 0 \) and \( \text{Re} \left\{ 1 + \mu + \nu \delta \left( \frac{b_j}{\beta_j} \right) \right\} > 0 \) \( (j = 1, \ldots, M) \) where \( M \) is an arbitrary positive integer.

(iii) \( M, N, P, Q \) are positive integers constrained by \( 1 \leq M \leq Q, 0 \leq N \leq P \).

(iv) \( | \arg (z) | < 1/2 \Omega \pi \).

(v) \( |(\beta - \alpha)(t - l)| < |\beta \{ w - l + \rho (t - l) + \sigma (w - t) \}|, \ t \in [a, b] \).

(vi) \( U \) is an arbitrary positive integer and the coefficients \( A_{V,R} \) \( (V,R=0) \) are arbitrary constants real or complex.

(vii) The series on the right hand side of (2.4) converges absolutely.

**APPLICATIONS**

(1) In this section we specifically show how the general integral formula (2.4) can be applied (and suitably maneuvered) to derive various interesting (and potentially useful) results including those given by Wille (1988) [9].

First of all \( \rho = \sigma = 0 \) and \( z = (\omega - l)^{(\gamma + \delta - 1)w} \), (2.4) readily reduces to

\[
\mathcal{R} = (w - l)^{-1} \sum_{R=0}^{\infty} \left( \frac{(-V)_{UR} A_{V,R}}{R!} \right) y^{\beta^R} \left( \frac{\lambda^\gamma}{\beta} \right)^R k^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{n} \frac{(-\eta)_e}{\eta!} \left( \frac{\alpha_1^R + \eta^R + \xi + \tau \epsilon}{h} \right)_{m+n} \beta^\eta \frac{(x^\xi)^\eta}{\eta!} \]

\[
\left( \beta \left( \frac{x^\xi}{\beta} \right) \right)^\eta \sum_{k=0}^{\infty} \frac{(-\eta)}{k!} (a_j, \alpha_j)_{1,N} (a_j, \alpha_j)_{N+1,P_i} (1 - \lambda, \nu) \left[ \begin{array}{c} \{ -\lambda - \mu - k - (\gamma + \delta) \Lambda - 1, v(\gamma + \delta) \} \\ \{ -\lambda - \mu - k - (\gamma + \delta) \Lambda - 1, v(\gamma + \delta) \} \end{array} \right] (2.5)
\]

provided that the conditions easily obtainable from those of (2.4) are satisfied.

Setting \( \beta = \alpha = \frac{1}{2} \) in (2.5) we obtain

\[
\mathcal{R} = \sum_{R=0}^{\infty} \left( \frac{(-V)_{UR} A_{V,R}}{R!} \right) y^{\beta^R} \left( \frac{\lambda^\gamma}{\beta} \right)^R k^{m+n} \sum_{\eta=0}^{m+n} \sum_{e=0}^{n} \frac{(-\eta)_e}{\eta!} \left( \frac{\alpha_1^R + \eta^R + \xi + \tau \epsilon}{h} \right)_{m+n} \beta^\eta \frac{(x^\xi)^\eta}{\eta!} \]

\[
I_{M,N}^{\eta} P_i, Q_i, M+2, \nu \left[ \begin{array}{c} \{ \nu^\nu \} \\ \{ \nu^\nu \} \end{array} \right] (1 - k - \lambda, \nu) \left( -k - \lambda - \gamma \Lambda, \gamma v \right) \left( -\mu - \delta \Lambda, \delta v \right) (1 - \Lambda, v) \left[ \begin{array}{c} (a_j, \alpha_j)_{1,N} (a_j, \alpha_j)_{N+1,P_i} \\ (1 - \Lambda, v) \{ -\lambda - \mu - k - (\gamma + \delta) \Lambda - 1, v(\gamma + \delta) \} \end{array} \right] (2.6)
\]

which in the further special case when \( l = 0, w = 1, r = 0, \gamma = \delta = \nu = 1, V = 0 \) and \( \alpha_j = 1 \) \( (j = 1, \ldots, P_i) \) \( \beta_j = 1 \) \( (j = 1, \ldots, Q_i) \) and reducing the generalized polynomial
set \( S_{n}^{\alpha',\beta',0}(x) \) to unity would yield one of Wille’s result [Wille1988[9],p.601.eq.(29)], on using Legendre’s Duplication formula for \( \Gamma \) function.

Next we put \( \gamma = \delta = 1, \lambda = \mu = -\frac{1}{2}, \alpha \to \alpha^2 \) and \( \beta \to \beta^2 \) in the integral formula (2.5), and sum the resulting series by mean of a known formula [Erdelyi et al. 1953[[1],p.101, eq.2.8(6)]: applying Legendre’s Duplication formula as well, we thus obtain the integral

\[
\int_{l}^{w} (t-l)^{-1/2} (w-t)^{-1/2} I_{P_i,Q_i,r}^{M,N} \left[ z \left\{ \frac{(t-l)(w-t)}{\alpha^2 (t-l) + \beta^2 (w-t)} \right\}^{\nu} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right. \\
\times S_{\nu}^{\alpha',\beta',0} \left[ x \left\{ \frac{(l-t)(w-l)}{\alpha^2 (l-t)+\beta^2 (w-t)} \right\}^{\eta} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right] dt \\
= \sqrt{\pi} \left[ \frac{V}{U} \right] \sum_{R=0}^{J} \frac{(-V)_{UR}A_{V,R}}{R!} \frac{y}{\beta^{2R}} \left( \frac{x}{\beta^2} \right)^R \frac{R'}{h^{m+n}} \sum_{\eta=0}^{m+n} \sum_{e=0}^{e} \frac{(-\eta)_e}{\eta!} \frac{(\alpha'+\eta+\xi+\tau \epsilon)}{h} \\
\times \left( \beta' \left( \frac{x}{\beta^2} \right)^{\eta} I_{P_i+1,Q_i+1,r}^{M,N+2} \left[ \left( \frac{(l-t)(w-l)}{\alpha(l-t)+\beta(w-t)} \right)^{\nu} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right] \right. \\
\left. \left( \frac{(l-t)(w-l)}{\alpha(l-t)+\beta(w-t)} \right)^{\eta} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right] \right] dt \\
= \sqrt{\pi} (w-l)^{-1} \sum_{R=0}^{J} \frac{(-V)_{UR}A_{V,R}2^{-1-\Lambda}}{R!} \frac{y}{\beta^{2R}} \left( \frac{x}{\beta^2} \right)^R \frac{R'}{h^{m+n}} \sum_{\eta=0}^{m+n} \sum_{e=0}^{e} \frac{(-\eta)_e}{\eta!} \frac{(\alpha'+\eta+\xi+\tau \epsilon)}{h} \\
\times \left( \frac{\alpha'+\eta+\xi+\tau \epsilon}{h} \right)^{\eta} \left( \frac{(l-t)(w-l)}{\alpha(l-t)+\beta(w-t)} \right)^{\eta} I_{P_i+2,Q_i+2,r}^{M,N+2} \left[ \left( \frac{4\alpha \beta}{\alpha+\beta} \right)^{-\nu} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right] \\
\left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \\
(1-\frac{1}{2}-\Lambda, \nu) \left( \frac{1}{2}-\frac{1}{2}\Lambda, \nu \right) \\
(2.7)
\]

which is the further special case when \( l = 0, w = 1, \nu = 1, V = 0 \) and \( \alpha_j = 1 \) (\( j = 1, ..., P_i \)) \( \beta_j = 1 \) (\( j = 1, ..., Q_i \)) immediately yields another result of Wille’s [Wille1988[9], p.601.eq.(29)].

If in our integral formula (2.5) we set \( \gamma = \delta = \frac{1}{2}, \mu = -\lambda - 2 \) and \( V \to 2V \), sum the resulting binomial series, and apply Legendre’s Duplication formula once again, we shall obtain

\[
\int_{l}^{w} (t-l)^{-1/2} (w-t)^{-1/2} I_{P_i,Q_i,r}^{M,N} \left[ \frac{(t-l)(w-t)}{\alpha^2 (t-l) + \beta^2 (w-t)} \right]^{\nu} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \\
\times S_{\nu}^{\alpha',\beta',0} \left[ \frac{(l-t)(w-l)}{\alpha^2 (l-t)+\beta^2 (w-t)} \right]^{\eta} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right] dt \\
= \sqrt{\pi} (w-l)^{-1} \sum_{R=0}^{J} \frac{(-V)_{UR}A_{V,R}2^{-1-\Lambda}}{R!} \frac{y}{\beta^{2R}} \left( \frac{x}{\beta^2} \right)^R \frac{R'}{h^{m+n}} \sum_{\eta=0}^{m+n} \sum_{e=0}^{e} \frac{(-\eta)_e}{\eta!} \frac{(\alpha'+\eta+\xi+\tau \epsilon)}{h} \\
\times \left( \frac{\alpha'+\eta+\xi+\tau \epsilon}{h} \right)^{\eta} \left( \frac{(l-t)(w-l)}{\alpha(l-t)+\beta(w-t)} \right)^{\eta} I_{P_i+2,Q_i+2,r}^{M,N+2} \left[ \left( \frac{4\alpha \beta}{\alpha+\beta} \right)^{-\nu} \left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \right] \\
\left( \frac{a_j, \alpha_j}{(b_j, \beta_j)_{M+1,Q_i}} \right) \\
(1-\frac{1}{2}-\Lambda, \nu) \left( \frac{1}{2}-\frac{1}{2}\Lambda, \nu \right) \\
(2.8)
\]
which hold true under the conditions readily obtainable stated with (2.4).

(2) By applying our integral in (2.4) to the case of Hermite polynomials (3, eq. (5.5.4), p. 106 and [4], p. 153) by $U = -2A_{V,R} = (-1)^R$ and reducing $S_n^{\alpha',\beta',0}[x]$ to unity we obtain:

$$\int_1^w \frac{(t-1)^{(w-t)^\mu}}{(t)^{(w-t)^{\mu+\nu}}i^n} I_{P,Q,r}^{M,N} \left[ z \{g(t)\}^v \right] \frac{(a_j, \alpha_j)_1, (a_j, \alpha_j)_{N+1,P_i}}{(b_j, \beta_j)_1, (b_j, \beta_j)_{M+1,Q_i}}$$

$$\times \left[ y \{g(t)\}^\rho \right] V/2 H_V \left[ \frac{1}{2y \{g(t)\}^\rho} \right] dt$$

$$= (w - l)^{-\Lambda - \alpha' \rho' R - 1} (1 + \sigma)^{-\mu - \delta R - 1} \sum_{R=0}^{\infty} \left( \frac{(-1)^R}{R!} \right) \sum_{k=0}^{\infty} \left( \frac{\beta - \alpha/\beta (1 + \rho)}{k!} \right)^k$$

$$\times I_{P+3,Q+2}^{M,N+3} \left[ z \{\beta (1 + \rho)^\gamma (1 + \sigma)^\delta\}^v \left( 1 - k - \rho' R, v \right) \left( -k - \lambda - \gamma \rho' R, v \right) \left( -\mu - \delta \rho' R, v \right) \left( 1 - \rho' R, v \right) \{\gamma - \mu - k - (\gamma + \delta) \rho' R - 1, v(\gamma + \delta) \} \right]$$

(2.9)

which holds true under the same conditions as those required for (2.4).

(3) If we reduce $S_n^U[x] \rightarrow L_n^\nu[x]$ (Laguerre polynomial) (3, eq. (5.16), p. 101 and [5], p. 158) and $S_n^{\alpha',\beta',0}[x]$ to Gould and Hopper Polynomial[7] by taking $U = 1$, $A_{V,R} = \left( \frac{n' + k'}{n'} \right) (k+1)$ and $q = m = \xi = 0$ and $h = -1$ in (2.4) we get

$$\int_1^w \frac{(t-1)^{(w-t)^\mu}}{(t)^{(w-t)^{\mu+\nu}}i^n} I_{P,Q,r}^{M,N} \left[ z \{g(t)\}^v \right] \frac{(a_j, \alpha_j)_1, (a_j, \alpha_j)_{N+1,P_i}}{(b_j, \beta_j)_1, (b_j, \beta_j)_{M+1,Q_i}}$$

$$\times L_n^{(k')^v} \left[ y \{g(t)\}^\rho \right] H_n^{(h)} \left[ y \{g(t)\}^\nu, \alpha', \beta' \right] dt$$

$$= (w - l)^{-\Lambda - \gamma \Lambda - 1} (1 + \sigma)^{-\mu - \delta \Lambda - 1} \sum_{R=0}^{\infty} \left( \frac{(-1)^R}{R!} \right) \sum_{k=0}^{\infty} \left( \frac{\beta - \alpha/\beta (1 + \rho)}{k!} \right)^k$$

$$\times \sum_{\eta=0}^{\infty} \sum_{e=0}^{\infty} \frac{(-\eta)_e}{\eta! e!} \left( \frac{\beta'}{\beta} \right)_t \eta \left( \frac{x}{\beta} \right)^\tau \left( \frac{y}{\nu} \right)_R \left( \frac{z}{\nu} \right)_R \left( -1 \right)_R$$

(2.10)

which holds true under the same conditions as those required for (2.4).
SPECIAL CASES

(i) On taking \( r = 1 \) and reducing \( S^U_n[x] \) and \( S_n^{\alpha',\beta',0}[x] \) to unity in (2.4) the result reduces to a known result derived by Srivastava H.M. and Raina R.K. ([5], p. 693, eq. 15).

(ii) By putting \( r = 1 \) and reducing \( S^U_n[x] \) and \( S_n^{\alpha',\beta',0}[x] \) to unity in (2.7) the result reduces to another result known result obtained by H.M. Srivastava and R.K. Raina ([5], p. 695, eq. 20 or 21).

ACKNOWLEDGEMENTS. The authors are grateful to Professor H.M. Srivastava, University of Victoria, Canada for his kind help and valuable suggestions in the preparation of this paper.

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Received: June, 2009