Existence of Positive Solutions for Nonlinear Singular Differential Systems Involving Parameter

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Abstract

In this paper, the authors investigate the existence of positive solutions for nonlinear singular boundary value problems for systems of differential equations involving parameter by using the fixed point theory in the cone.

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1. Introduction

In this paper, we consider the existence of positive solutions with the following system of second-order singular ordinary differential equations:
\[ x''(t) + k^2 x(t) = \lambda f(t, x(t), y(t)), \quad 0 < t < 1, \]
\[ y''(t) + k^2 y(t) = \lambda g(t, x(t), y(t)), \]
\[ x'(0) = x'(1) = 0, \]
\[ y'(0) = y'(1) = 0, \]
\[ (1.1) \]

where \( \lambda > 0 \) is a parameter, \( k \in (0, \pi/2) \) is a constant, the nonlinearities \( f, g \) are continuous functions and may have singularity at \( t = 0, 1. \)

There has been increasing interest in the subject of singular differential equations system due to its strongly application background, to identify a few, we refer the reader to see[1-4]. Recently, Agarwal and O’Regan[3] studied the equation \( x''(t) + p(t)f(t, x) = 0 \) with the boundary condition \( x(0) = x(1) = 0, \) they obtained that the equation has at least one positive solution under some restrictions on the nonlinear term \( f(t, x). \) Sun et al.[4] established the existence theorems of positive solutions for the following equation \( x'' + k^2 x = f(t, x) \) and \( x'(0) = x'(1) = 0, \) where \( k \in (0, \pi/2) \) is a constant, the nonlinear terms \( f(t), g(t, x) \) are continuous functions and may have singularity at \( t = 0, 1. \)

Motivated by the work of above papers, the aim of this paper is to consider the existence of positive solutions to the singular system (1.1). The problem we discuss is different from those in [3, 4]. Firstly, which we discuss is the system instead of the individual equation. Secondly, the system (1.1) involves a parameter \( \lambda. \) Finally, the techniques used in this paper are the approximation method, in order to overcome the difficulties caused by singularity and to apply the fixed point theorem in cone.

2. Preliminaries and some lemmas

Let \( G(t, s) \) be the Green’s function for the following boundary value problem
\[ x''(t) + k^2 x(t) = 0, \quad 0 < t < 1, \]
\[ x'(0) = x'(1) = 0, \]
Existence of positive solutions

that is

\[
G(t, s) = \begin{cases} 
\frac{1}{k \sin k} \cos k(1 - t) \cos k s, & 0 \leq s \leq t \leq 1, \\
\frac{1}{k \sin k} \cos k(1 - s) \cos k t, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

(2.1)

Remark 2.1 From (2.2), it is easy to get the following properties of \(G(t, s)\):

1. \(G(t, s) \geq 0\) for any \((t, s) \in [0, 1] \times [0, 1]\).
2. \(G(t, s) > 0\) for any \((t, s) \in (0, 1) \times (0, 1)\).
3. \(G(t, s) \geq \frac{\cos^2 k}{k \sin k} \) for any \((t, s) \in [0, 1] \times [0, 1]\).
4. \(G(t, s) \leq \frac{1}{k \sin k} \) for any \((t, s) \in [0, 1] \times [0, 1]\).
5. \(\cos^2 k G(s, s) \leq G(t, s) \leq \frac{1}{\cos^2 k} G(s, s), \) \((t, s) \in [0, 1] \times [0, 1]\).

This paper, the basic space used is \(X = C^+ [0, 1] \times C^+ [0, 1]\), obviously, it is the Banach space with the norm \(\| (x, y) \| = \| x \| + \| y \| \) where \(\| x \| = \max_{t \in [0, 1]} |x(t)|, \) \(\| y \| = \max_{t \in [0, 1]} |y(t)|\), Define

\[
K = \left\{ (x, y) \in X : \min_{t \in [0, 1]} x(t) \geq \cos^2 k \| x \|, \min_{t \in [0, 1]} y(t) \geq \cos^2 k \| y \| \right\}.
\]

Definition 2.1 The vector \((x, y) \in C^1 [0, 1] \cap C^2 [0, 1]\) is said to be a positive solution of the system (1.1) if and only if \((x, y)\) satisfies system (1.1) and \(x(t) > 0, \) \(y(t) \geq 0\) or \(x(t) \geq 0, \) \(y(t) > 0\) for any \(t \in (0, 1)\).

Lemma 2.1[5] Let \(P\) be a positive cone in a real Banach space. Denote \(P_r, P_r, R = \{ x \in P : \| x \| = \min_{t \in [0, 1]} x(t) \geq \cos^2 k \| x \| \}\), \(0 < r < R < +\infty\). Let \(A : P_{r, R} \rightarrow P\) be a completely continuous operator. If the following conditions are satisfied:

1. \(\| Ax \| \leq \| x \|,\) \(\forall x \in \partial P_r.\)
2. there exists a \(x_0 \in \partial P_1,\) such that \(x \neq Ax + mx_0,\) \(\forall x \in \partial P_r,\) \(m > 0.\)

then \(A\) has fixed points in \(P_{r, R.}\)

Remark 2.2 If (1) and (2) are satisfied for \(x \in \partial P_r\) and \(x \in \partial P_R\) respectively. Then Lemma 2.1 is still true.

3. Main results

Let us list the following assumptions:
The functions \( f, g : (0, 1) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) are continuous functions and satisfy \( f(t, u, v) \leq \phi_1(t)h_1(t, u, v), g(t, u, v) \leq \phi_2(t)h_2(t, u, v) \) for any \((t, u, v) \in (0, 1) \times [0, +\infty) \times [0, +\infty), \) where \( \phi_i : (0, 1) \to [0, +\infty) \) is continuous and singular at \( t = 0, \phi_i(t) \neq 0 \) on \([0, +\infty), h_i : [0, 1] \times [0, +\infty) \times [0, +\infty) \to [0, +\infty) \) is continuous, \((i = 1, 2).\)

\((H_2)\) \( 0 < \int_0^1 G(s, s)\phi_i(s)ds < +\infty, \) \((i = 1, 2).\)

From the above assumptions \((H_1) (H_2), \) let

\[
A(x, y)(t) = \lambda \int_0^1 G(t, s)f(s, x(s), y(s))ds,
\]

\[
B(x, y)(t) = \lambda \int_0^1 G(t, s)g(s, x(s), y(s))ds,
\]

\[
F(x, y) = (A(x, y), B(x, y)), \quad t \in (0, 1).
\]

Obviously the system \((1.1)\) has a solution \((x, y)\) if and only if \((x, y) \in K\) is a fixed point of the operator \( F \) defined by \((3.1), \) i.e. \( F(x, y) = (x, y).\)

**Theorem 3.1** Assume \((H_1)(H_2)\) hold, \( F : K \to K \) is completely continuous.

**Proof** First we show that \( F : K \to K \) is well defined. Let \((x, y) \in K, t \in [0, 1].\) From \((H_1),\) it is easy to see that \( F(x, y) \geq (0, 0).\) For any \( t \in [0, 1], \)

\[
A(x, y)(t) = \lambda \int_0^1 G(t, s)f(s, x(s), y(s))ds \leq \frac{\lambda}{\cos^2 k} \int_0^1 G(s, s)f(s, x(s), y(s))ds.
\]

That is \( \|A(x, y)\| \leq \frac{\lambda}{\cos^2 k} \int_0^1 G(s, s)f(s, x(s), y(s))ds.\) Also, we have

\[
A(x, y)(t) = \lambda \int_0^1 G(t, s)f(s, x(s), y(s))ds \geq \cos^2 k \|A(x, y)\|.
\]

In the same way, we can show that \( \|B(x, y)\| \geq \cos^2 k \|B(x, y)\|.\) Hence, \( F(x, y) \) is well defined for any \((x, y) \in K\) and \( F : K \to K.\)

Next, for any positive integer \( m, \) we define an operator \( F_m : K \to K \) by

\[
A_m(x, y)(t) = \lambda \int_{\frac{1}{m}}^1 G(t, s)f(s, x(s), y(s))ds,
\]

\[
B_m(x, y)(t) = \lambda \int_{\frac{1}{m}}^1 G(t, s)g(s, x(s), y(s))ds,
\]

\[
F_m(x, y) = (A_m(x, y), B_m(x, y)), \quad t \in [0, 1].
\]
It is easy to prove that $F_m : K \to K$ is completely continuous, for each $m \geq 1$.

Finally, we show that $F : K \to K$ is a completely continuous operator. For any $t \in [0, 1]$ and $(x, y) \in K$, we have

$$|A(x, y)(t) - A_m(x, y)(t)| = \lambda \int_0^t G(t, s) f(s, x(s), y(s)) ds \leq \frac{\lambda}{\cos^2 k} \int_0^t G(s, s) \phi_1(s) h_1(s, x(s), y(s)) ds < +\infty.$$ 

This together with $(H_2)$ implies that

$$\|A(x, y) - A_m(x, y)\| \leq \frac{\lambda}{\cos^2 k} \int_0^t G(s, s) \phi_1(s) h_1(s, x(s), y(s)) ds \to 0, \ m \to +\infty.$$ 

By the same proof, we obtain that $\|B(x, y) - B_m(x, y)\| \to 0, \ m \to +\infty$. Therefore $F : K \to K$ completely continuous.

**Theorem 3.2** Assume that $(H_1)$ $(H_2)$ hold, $h_1, h_2$ and $f$ satisfy the following condition:

$$(H_3) \ 0 \leq h_1^0 = \limsup_{u \to 0+} \max_{t \in [0, 1]} \frac{h_1(t, u, v)}{u}, \quad h_2^0 = \limsup_{v \to 0+} \max_{t \in [0, 1]} \frac{h_2(t, u, v)}{v} < L,$$

and $0 < l < f_\infty = \liminf_{u \to +\infty} \min_{t \in [a, b] \subset [0, 1]} \frac{f(t, u, v)}{u} \leq +\infty,$

where $L = \left(\max\left\{\frac{1}{\cos^2 k} \int_0^1 G(s, s) \phi_1(s) ds, \frac{1}{\cos^2 k} \int_0^1 G(s, s) \phi_2(s) ds\right\}\right)^{-1}, \ l = \left(f_a^0 G(t, s) ds\right)^{-1}, \ t \in [a, b].$

Then system (1.1) has at least one positive solution for any $\lambda \in \left(\frac{l}{f_\infty}, \frac{L}{\max(h_1^0, h_2^0)}\right).$

**Proof** From the condition, there exists $\varepsilon > 0$, such that $\frac{l}{f_\infty - \varepsilon} \leq \lambda \leq \frac{L}{\max(h_1^0, h_2^0) + \varepsilon}$. By the first inequality of $(H_3)$, there exists $r > 0$, for the above $\varepsilon$, such that $h_1(t, u, v) \leq (h_1^0 + \varepsilon) u, \ h_2(t, u, v) \leq (h_2^0 + \varepsilon) v, \ 0 \leq u, v \leq r, \ t \in [0, 1].$

Set $K_{r_1} = \{(x, y) \in K : \|x\| < r_1, \ \|y\| < r_1\}, \ (r_1 \leq r)$, by the definition of $\|\cdot\|$, we know that $x(t) \leq \|x\| = r_1 \leq r, \ y(t) \leq \|y\| = r_1 \leq r, \ \forall (x, y) \in$
\[ \partial K_{r_1}, \ t \in [0, 1]. \] Then for any \((x, y) \in \partial K_{r_1}, \ t \in [0, 1],\]

\[
\|A(x, y)\| = \lambda \max_{t \in [0, 1]} \left| \int_0^1 G(t, s)f(s, x(s), y(s))ds \right| \\
\leq \lambda \max_{t \in [0, 1]} \int_0^1 G(t, s)\phi_1(s)h_1(s, x(s), y(s))ds \\
\leq \lambda \max_{t \in [0, 1]} \int_0^1 G(t, s)\phi_1(s)(h_1^0 + \varepsilon)x(s)ds \\
\leq \max_{s \in [0, 1]} x(s)\lambda(h_1^0 + \varepsilon) \frac{1}{\cos^2 k} \int_0^1 G(s, s)\phi_1(s)ds \leq r_1 = \|x\|. \\
\]

Similarly, we have \(\|B(x, y)\| \leq r_1 = \|y\|. \) Thus,

\[
\|F(x, y)\| = \|A(x, y)\| + \|B(x, y)\| \leq \|x\| + \|y\| = \|(x, y)\|. \quad (3.2)
\]

On the other hand, by the second inequality of \((H_3)\), there exists \(r_0 > \cos^2 kr_1 > 0\), for the above \(\varepsilon\) such that

\[
f(t, u, v) \geq (f_\infty - \varepsilon)u, \ u \geq r_0, \ t \in [a, b] \subset [0, 1]. \quad (3.3)
\]

Write \(r_2 \geq r_0/\cos^2 k > r_1, \ \ K_{r_2} = \{(x, y) \in K : \|x\| < r_2, \ \|y\| < r_2\}\). Let \((x_0, y_0) = (1, 1) \in \partial K_1 = \{(x, y) \in K : \|x\| = 1, \ \|y\| = 1\}\), then

\[
(x, y) \neq F(x, y) + \mu(x_0, y_0), \ \forall (x, y) \in \partial K_{r_2}, \ \forall \mu > 0. \quad (3.4)
\]

Suppose that \((3.4)\) were false, then there exists \((x_0, y_0) = (1, 1) \in \partial K_1, \ (x_2, y_2) \in \partial K_{r_2}, \ \mu_2 > 0, \) such that \((x_2, y_2) = F(x_2, y_2) + \mu_2(x_0, y_0)\).

\((3.3)\) and the fact \(x_2(t) \geq \cos^2 k\|x_2\| = \cos^2 kr_2 \geq r_0, \ t \in [a, b] \subset [0, 1]\), tell us that \(f(t, x_2(t), y_2(t)) \geq (f_\infty - \varepsilon)x_2(t), \ t \in [a, b] \subset [0, 1]\).

Let \(\sigma = \min\{x_2(t) : t \in [a, b]\}\), for \((u_2, v_2) \in K_{r_2}, \ t \in [a, b]\), we have

\[
x_2(t) = \lambda \int_0^1 G(t, s)f(s, x_2(s), y_2(s))ds + \mu_0 \\
\geq \lambda \int_a^b G(t, s)(f_\infty - \varepsilon)x_2(s)ds + \mu_0 \\
\geq \lambda(f_\infty - \varepsilon)\sigma \int_a^b G(t, s)ds + \mu_0 \\
\geq \sigma + \mu_0 > \sigma,
\]
which is a contradicts, that implies that (3.4) holds.

From (3.2), (3.4), Lemma 2.2, theorem 3.1 and the fact that $\overline{K}_{r_1} \subset K_{r_2}$, we can obtain that the operator $F$ has fixed point $(x, y)$, which belongs to $K_{r_2} \setminus \overline{K}_{r_1}$, such that $0 < 2r_1 < \| (x, y) \| < 2r_2$. It is easy to see that $(x, y)$ is a positive solution of system (1.1).

**Remark 3.1** Noticing that $\frac{1}{f_\infty} < 1$, $\frac{t}{g_\infty - k^2} < 1$ and $\frac{L}{h_1^0} > 1$, $\frac{L}{h_2^0} > 1$, so if $\lambda = 1$, Theorem 3.2 also holds.

**Remark 3.2** From Theorem 3.2, we can see $h_1(t, x, y)$, $h_2(t, x, y)$ and $f(t, x, y)$ need not be superlinear or sublinear. In fact, Theorem 3.2 still holds, if one of the following conditions is satisfied

1. If $f_\infty = +\infty$, $h_1^0 > 0$, $h_2^0 > 0$, then for each $\lambda \in (0, \frac{L}{\max\{h_1^0, h_2^0\}})$.
2. If $f_\infty = +\infty$, $h_1^0 = h_2^0 = 0$, then for each $\lambda \in (0, +\infty)$.
3. If $f_\infty > l > 0$, $h_1^0 = h_2^0 = 0$, then for each $\lambda \in (\frac{l}{f_\infty}, +\infty)$.

**Remark 3.3** If we replace the second inequality of $(H_3)$: $0 < l < f_\infty = \lim\inf_{u \to +\infty} \min_{t \in [0,1]} \frac{f(t, u, v)}{u} \leq +\infty$ by $0 < l < g_\infty = \lim\inf_{v \to +\infty} \min_{t \in [0,1]} \frac{g(t, u, v)}{v} \leq +\infty$, when $\lambda \in \left(\frac{l}{g_\infty}, \frac{L}{\max\{h_1^0, h_2^0\}}\right)$, Theorem 3.2 also holds.

**Theorem 3.3** Assume that $(H_1)$ $(H_2)$ hold, $h_1$, $h_2$ and $f$ satisfy the following condition:

$$(H_4) \quad 0 \leq h_1^\infty = \lim\sup_{u \to +\infty} \max_{t \in [0,1]} \frac{h_1(t, u, v)}{u}, \quad h_2^\infty = \lim\sup_{v \to +\infty} \max_{t \in [0,1]} \frac{h_2(t, u, v)}{v} < L,$$

and $0 < l < f_0 = \lim\inf_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t, u, v)}{u} \leq +\infty$.

where constants $L$ and $l$ are defined by Theorem 3.2.

Then system (1.1) has at least one positive solution for any $\lambda \in \left(\frac{l}{f_0}, \frac{L}{\max\{h_1^0, h_2^0\}}\right)$.

**Proof** The proof is similar to that of Theorem 3.2, and so we omit it.

**Remark 3.4** Noticing that $\frac{1}{f_0} < 1$ and $\frac{L}{h_1^\infty} > 1$, $\frac{L}{h_2^\infty} > 1$, so if $\lambda = 1$, Theorem 3.3 also holds.

**Remark 3.5** We can see that Theorem 3.3 still holds, if one of the following conditions is satisfied

1. If $h_1^\infty < L$, $h_2^\infty < L$, $f_0 = +\infty$, then for each $\lambda \in (0, \frac{L}{\max\{h_1^0, h_2^0\}})$.
2. If $h_1^\infty = h_2^\infty = 0$, $f_0 = +\infty$, then for each $\lambda \in (0, +\infty)$.
3. If $h_1^\infty = h_2^\infty = 0$, $f_0 > l > 0$, then for each $\lambda \in (\frac{l}{f_0}, +\infty)$.

**Remark 3.6** If we replace the second inequality of $(H_4)$: $0 < l < f_0 =
\[
\lim_{u \to 0^+} \min_{t \in [a,b] \subset [0,1]} \frac{f(t,u,v)}{u} \leq +\infty \text{ by } 0 < l < g_0 = \lim_{v \to 0^+} \min_{t \in [a,b] \subset [0,1]} \frac{g(t,u,v)}{v} \leq +\infty, \text{ when } \lambda \in \left(\frac{l}{g_0}, \frac{L}{\max\{h_1^{-1}, h_2^{-1}\}}\right), \text{ Theorem 3.3 also holds.}
\]

References


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