On Separation Axioms in Biminimal Structure Spaces

Maliwan Tunapan, Chokchai Viriyapong, Witchaya Rattanametawee and Chawalit Boonpok

Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham 44150, Thailand
chawalit.b@msu.ac.th

Abstract

The aim of this paper is to introduce the concepts of pairwise $\Lambda_m$-$T_0$, pairwise $\Lambda_m$-$T_1$ and pairwise $\Lambda_m$-$T_2$ biminimal structure spaces and define pairwise $\Lambda_m$-$R_{i-1}$ biminimal structure spaces for $i = 1, 2$. We study some of the fundamental properties of such spaces. Moreover, we investigate their relationship to some other known separation axioms.

Mathematics Subject Classification: 54B05, 54C08

Keywords: minimal structure, $m$-space, biminimal structure space, $(i, j)$-$(\Lambda, m)$-closed set, pairwise $\Lambda_m$-$T_0$, pairwise $\Lambda_m$-$T_1$, pairwise $\Lambda_m$-$T_2$, pairwise $\Lambda_m$-$R_0$, pairwise $\Lambda_m$-$R_1$

1 Introduction

The separation axioms $R_0$ and $R_1$ in topological spaces was introduced by Shanin [21] in 1943. Later, Davis [7] rediscovered it and studied some properties of this weak separation axiom. Several topology (e.g. [10], [18]) further investigated properties of $R_0$ topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of $R_1$ topological spaces which are independent of both $T_0$ and $T_1$ but strictly weaker than $T_2$. Murdeshwar and Naimpally [17] and Dube [11] studied some of the fundamental properties of the class of $R_1$ topological spaces. As natural generalizations of the separation axioms $R_0$ and $R_1$, the concepts of semi-$R_0$ and semi-$R_1$ were introduced by Murdeshwar and Prasad
[14] and Dorsett [9]. In [5] the concepts of the \((\Lambda, \theta)\)-closure and \((\Lambda, \theta)\)-open sets were introduced by using \(\theta\)-open sets and \(\theta\)-closure operations due to Velicko [22]. Maki [15] called a subset \(A\) of a topological space \((X, \tau)\) a \(\Lambda\)-set if it is the intersection of open sets containing \(A\). Arenus et al. [1] defined a subset \(A\) to be \(\lambda\)-closed if \(A = L \cap F\), where \(L\) is a \(\Lambda\)-set and \(F\) is a closed in \((X, \tau)\). They used \(\lambda\)-closed sets to characterize some low separation axioms. Caldas and Dontchev [3] introduced the notions of \(\Lambda_s\)-sets and generalized \(\Lambda_s\)-sets and obtained a characterization of semi-\(T_{1\frac{1}{2}}\)-spaces. Dontchev and Maki [8] introduced semi-\(\lambda\)-closed sets and used them to obtain a decomposition of quasi-continuity. Cammaroto and Noiri [6] defined \(\Lambda_m\)-sets, \((\Lambda, m)\)-closed sets and generalized \(\Lambda_m\)-sets in an \(m\)-space \((X, m_X)\) which is equivalent to a generalized topological space [13] and investigate properties of several low separation axioms of topologies constructed by the families of these sets. Caldas et al. [4] introduced the notions of \(\Lambda_\delta-T_0\), \(\Lambda_\delta-T_1\), \(\Lambda_\delta-R_0\) and \(\Lambda_\delta-R_1\) topological spaces as a version of the known notions of \(R_0\) and \(R_1\) topological spaces.

Boonpok [2] introduced the notion of biminimal structure spaces and studied \(m^1_X m^2_X\)-closed sets and \(m^1_X m^2_X\)-open sets in biminimal structure spaces.

In this paper, we introduce the notions of pairwise \(\Lambda_m-T_0\), pairwise \(\Lambda_m-T_1\), pairwise \(\Lambda_m-R_0\) and pairwise \(\Lambda_m-R_1\) in biminimal structure spaces and investigate their fundamental properties.

2 Preliminaries

**Definition 2.1.** [19] Let \(X\) be a nonempty set and \(\mathcal{P}(X)\) the power set of \(X\). A subfamily \(m_X\) of \(\mathcal{P}(X)\) is called a minimal structure (briefly \(m\)-structure) on \(X\) if \(\emptyset \in m_X\) and \(X \in m_X\).

By \((X, m_X)\), we denote a nonempty set \(X\) with an \(m\)-structure \(m_X\) on \(X\) and it is called an \(m\)-space. Each member of \(m_X\) is said to be \(m_X\)-open and the complement of an \(m_X\)-open set is said to be \(m_X\)-closed.

**Definition 2.2.** [19] Let \(X\) be a nonempty set and \(m_X\) an \(m\)-structure on \(X\). For a subset \(A\) of \(X\), the \(m_X\)-closure of \(A\) and the \(m_X\)-interior of \(A\) are defined as follows:

1. \(m_X-\text{Cl}(A) = \cap\{F : A \subseteq F, X - F \in m_X\}\),
2. \(m_X-\text{Int}(A) = \cup\{U : U \subseteq A, U \in m_X\}\).

**Lemma 2.3.** [16] Let \(X\) be a nonempty set and \(m_X\) a minimal structure on \(X\). For subsets \(A\) and \(B\) of \(X\), the following properties hold:

1. \(m_X-\text{Cl}(X - A) = X - (m_X-\text{Int}(A))\) and \(m_X-\text{Int}(X - A) = X - (m_X-\text{Cl}(A))\).
(2) If \((X - A) \in m_X\), then \(m_X\text{-}Cl(A) = A\) and if \(A \in m_X\), then \(m_X\text{-}Int(A) = A\).

(3) \(m_X\text{-}Cl(\emptyset) = \emptyset\), \(m_X\text{-}Cl(X) = X\), \(m_X\text{-}Int(\emptyset) = \emptyset\) and \(m_X\text{-}Int(X) = X\).

(4) If \(A \subseteq B\), then \(m_X\text{-}Cl(A) \subseteq m_X\text{-}Cl(B)\) and \(m_X\text{-}Int(A) \subseteq m_X\text{-}Int(B)\).

(5) \(A \subseteq m_X\text{-}Cl(A)\) and \(m_X\text{-}Int(A) \subseteq A\).

(6) \(m_X\text{-}Cl(m_X\text{-}Cl(A)) = m_X\text{-}Cl(A)\) and \(m_X\text{-}Int(m_X\text{-}Int(A)) = m_X\text{-}Int(A)\).

**Lemma 2.4.** [16] Let \(X\) be a nonempty set with a minimal structure \(m_X\) and \(A\) a subset of \(X\). Then \(x \in m_X\text{-}Cl(A)\) if and only if \(U \cap A \neq \emptyset\) for every \(U \in m_X\) containing \(x\).

**Definition 2.5.** [16] An \(m\)-structure \(m_X\) on a nonempty set \(X\) is said to have **property** \(B\) if the union of any family of subsets belonging to \(m_X\) belongs to \(m_X\).

**Lemma 2.6.** [19] Let \(X\) be a nonempty set and \(m_X\) an \(m\)-structure on \(X\) satisfying property \(B\). For a subset \(A\) of \(X\), the following properties hold:

(1) \(A \in m_X\) if and only if \(m_X\text{-}Int(A) = A\),

(2) If \(A\) is \(m_X\)-closed if and only if \(m_X\text{-}Cl(A) = A\),

(3) \(m_X\text{-}Int(A) \in m_X\) and \(m_X\text{-}Cl(A)\) is \(m_X\)-closed.

**Definition 2.7.** [2] Let \(X\) be a nonempty set and let \(m_X^1, m_X^2\) be minimal structures on \(X\). A triple \((X, m_X^1, m_X^2)\) is called a **biminimal structure space** (briefly bim-space).

Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and let \(A\) be a subset of \(X\). The \(m_X\)-closure and \(m_X\)-interior of \(A\) with respect to \(m_X^i\) are denoted by \(m_X^i\text{-}Cl(A)\) and \(m_X^i\text{-}Int(A)\), respectively, for \(i = 1, 2\).

### 3 \((i,j)\)-\(\Lambda_m\)-open sets and associated separation axioms

**Definition 3.1.** Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A\) a subset of \(X\). Then the \(i\)-\(\Lambda_m(A)\) is define as follows: \(i\text{-}\Lambda_m(A) = \cap\{U : A \subseteq U, U \in m_X^i\}\) for \(i = 1, 2\). A subset \(A\) of \(X\) is called a \(i\text{-}\Lambda_m\)-set if \(A = i\text{-}\Lambda_m(A)\).

The family of all \(i\text{-}\Lambda_m\)-sets of \((X, m_X^1, m_X^2)\) is denoted by \(i\text{-}\Lambda_m(X)\).
Definition 3.2. A subset $A$ of a biminimal structure space $(X, m_X^1, m_X^2)$ is said to be $(i, j)$-$(\Lambda, m)$-closed if $A = U \cap F$, where $U$ is an $i$-$\Lambda_m$-set and $F$ is a $m_X^j$-closed set, where $i, j = 1, 2$ and $i \neq j$. The complement of a $(i, j)$-$(\Lambda, m)$-closed set is called $(i, j)$-$(\Lambda, m)$-open.

The family of all $(i, j)$-$(\Lambda, m)$-open (resp. $(i, j)$-$(\Lambda, m)$-closed) sets of $(X, m_X^1, m_X^2)$ is denoted by $(i, j)$-$(\Lambda, m)$-open (resp. $(i, j)$-$(\Lambda, m)$-closed) sets of $(X, m_X^1, m_X^2)$.

Definition 3.3. Let $A$ be a subset of a biminimal structure space $(X, m_X^1, m_X^2)$. A point $x \in A$ is called a $(i, j)$-$(\Lambda, m)$-cluster point of $A$ if for every $(i, j)$-$(\Lambda, m)$-open set $U$ of $(X, m_X^1, m_X^2)$ containing $x$, $A \cap U \neq \emptyset$, where $i, j = 1, 2$ and $i \neq j$. The set of all $(i, j)$-$(\Lambda, m)$-cluster points is called the $(i, j)$-$(\Lambda, m)$-closure of $A$ and is denoted by $(i, j)$-$\Lambda_m C(X)$.

Lemma 3.4. Let $A$ and $B$ be subsets of a biminimal structure space $(X, m_X^1, m_X^2)$, where $m_X^1, m_X^2$ have property $B$. For the $(i, j)$-$(\Lambda, m)$-closure, where $i, j = 1, 2$ and $i \neq j$, the following properties hold.

1. $A \subseteq (i, j)$-$\Lambda_m A$.
2. $(i, j)$-$\Lambda_m A = \cap \{ F : F \subseteq A, F \in (i, j)$-$\Lambda_m C(X) \}$.
3. If $A \subseteq B$, then $(i, j)$-$\Lambda_m A \subseteq (i, j)$-$\Lambda_m B$.
4. $A$ is $(i, j)$-$(\Lambda, m)$-closed if and only if $(i, j)$-$\Lambda_m A = A$.
5. $(i, j)$-$\Lambda_m A$ is $(i, j)$-$(\Lambda, m)$-closed.

Lemma 3.5. Let $A$ be a subset of a biminimal structure space $(X, m_X^1, m_X^2)$, where $m_X^1, m_X^2$ have property $B$. Then the following properties hold.

1. If $A_\beta$ is $(i, j)$-$(\Lambda, m)$-closed for each $\beta \in \Delta$, then $\cap_{\beta \in \Delta} A_\beta$ is $(i, j)$-$(\Lambda, m)$-closed, where $i, j = 1, 2$ and $i \neq j$.
2. If $A_\beta$ is $(i, j)$-$(\Lambda, m)$-open for each $\beta \in \Delta$, then $\cup_{\beta \in \Delta} A_\beta$ is $(i, j)$-$(\Lambda, m)$-open, where $i, j = 1, 2$ and $i \neq j$.

Definition 3.6. Let $A$ be a subset of a biminimal structure space $(X, m_X^1, m_X^2)$. Then the $(i, j)$-$\Lambda_m$-kernel of $A$, denoted by $(i, j)$-$\Lambda_m Ker(A)$, is defined to be the set $(i, j)$-$\Lambda_m Ker(A) = \cap \{ G : A \subseteq G, G \in (i, j)$-$\Lambda_m O(X) \}$, where $i, j = 1, 2$ and $i \neq j$.

Lemma 3.7. For any two subsets $A, B$ of a biminimal structure space $(X, m_X^1, m_X^2)$. Then the following properties hold.

1. If $A \subseteq B$, then $(i, j)$-$\Lambda_m Ker(A) \subseteq (i, j)$-$\Lambda_m Ker(B)$, where $i, j = 1, 2$ and $i \neq j$. 
Remark 1. If a biminimal structure space

\[ (i, j) \Lambda_m \text{Ker}((i, j) \Lambda_m \text{Ker}(A)) = (i, j) \Lambda_m \text{Ker}(A) \], where \( i, j = 1, 2 \) and \( i \neq j \).

Lemma 3.8. For any two points \( x, y \) of a biminimal structure space \( (X, m^1_X, m^2_X) \),
\( y \in (i, j) \Lambda_m \text{Ker}(\{x\}) \) if and only if \( x \in (i, j) \{y\}^{\Lambda,m} \), where \( i, j = 1, 2 \) and \( i \neq j \).

Proof. Let \( y \not\in (i, j) \Lambda_m \text{Ker}(\{x\}) \). Then, there exists a \((i, j)-(\Lambda, m)\)-open set \( V \) containing \( x \) such that \( y \not\in V \). Hence, \( x \not\in (i, j) \{y\}^{\Lambda,m} \). The converse is similarly shown. \( \square \)

Definition 3.9. A subset \( N_x \) of a biminimal structure space \( (X, m^1_X, m^2_X) \) is called a \((i, j)-(\Lambda, m)\)-neigbourhood of a point \( x \in X \) if there exists a \((i, j)-(\Lambda, m)\)-open set \( U \) such that \( x \in U \subseteq N_x \), where \( i, j = 1, 2 \) and \( i \neq j \).

Proposition 3.10. Let \( A \) be a subset of a biminimal structure space \( (X, m^1_X, m^2_X) \). Then \((i, j)-\Lambda_m \text{Ker}(A) = \{x \in X : (i, j)-\{x\}^{\Lambda,m} \cap A \neq \emptyset\} \), where \( i, j = 1, 2 \) and \( i \neq j \).

Proof. Let \( x \in (i, j)-\Lambda_m \text{Ker}(A) \) and suppose that \((i, j)-\{x\}^{\Lambda,m} \cap A = \emptyset\). Then \( x \not\in X - ((i, j)-\{x\}^{\Lambda,m}) \) which is a \((i, j)-(\Lambda, m)\)-open set containing \( A \). But \( x \in (i, j)-\Lambda_m \text{Ker}(A) \). Consequently, \((i, j)-\{x\}^{\Lambda,m} \cap A \neq \emptyset\). Let \( x \in X \) such that \( \{x\}^{\Lambda,m} \cap A \neq \emptyset \) and suppose that \( x \not\in (i, j)-\Lambda_m \text{Ker}(A) \). Then, there exists a \((i, j)-(\Lambda, m)\)-open set \( U \) containing \( A \) and \( x \not\in U \). Let \( y \in (i, j)-\{x\}^{\Lambda,m} \cap A \). Hence, \( U \) is a \((i, j)-(\Lambda, m)\)-neighbourhood of \( y \) which does not contain \( x \). By this contradiction \( x \in (i, j)-\Lambda_m \text{Ker}(A) \). \( \square \)

Definition 3.11. A biminimal structure space \( (X, m^1_X, m^2_X) \) is said to be:

(a) \textit{pairwise} \( \Lambda_m T_0 \) if for each pair of distinct points in \( X \), there exists a \((i, j)-(\Lambda, m)\)-open set containing one of the points but not the other, where \( i, j = 1, 2 \) and \( i \neq j \).

(b) \textit{pairwise} \( \Lambda_m T_1 \) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exists a \((i, j)-(\Lambda, m)\)-open set \( U \) in \( X \) containing \( x \) but not \( y \) and a \((j, i)-(\Lambda, m)\)-open set \( V \) in \( X \) containing \( y \) but not \( x \), where \( i, j = 1, 2 \) and \( i \neq j \).

(c) \textit{pairwise} \( \Lambda_m T_2 \) if for each pair of distinct points \( x \) and \( y \) in \( X \), there exist a \((i, j)-(\Lambda, m)\)-open set \( U \) and \((j, i)-(\Lambda, m)\)-open set \( V \) such that \( x \in U, y \in V \) and \( U \cap V = \emptyset \), where \( i, j = 1, 2 \) and \( i \neq j \).

Remark 1. If a biminimal structure space \((X, m^1_X, m^2_X)\) is pairwise \( \Lambda_m T_i \), then it is pairwise \( \Lambda_m T_{i-1} \) for \( i = 1, 2 \).
Theorem 3.12. A biminimal structure space \( (X, m^1_X, m^2_X) \) is pairwise \( \Lambda_m-T_0 \) if and only if, for each pair of distinct points \( x, y \) of \( X \), \((i, j)-\{x\}^{(\Lambda,m)} \neq (j, i)-\{y\}^{(\Lambda,m)}\), where \( i, j = 1, 2 \) and \( i \neq j \).

Proof. Suppose that \( x, y \in X \), \( x \neq y \) and \((i, j)-\{x\}^{(\Lambda,m)} \neq (j, i)-\{y\}^{(\Lambda,m)}\). Let \( z \) be a point of \( X \) such that \( z \in (i, j)-\{x\}^{(\Lambda,m)} \) but \( z \not\in (j, i)-\{y\}^{(\Lambda,m)} \). We claim that \( x \not\in (j, i)-\{y\}^{(\Lambda,m)} \). If \( x \in (j, i)-\{y\}^{(\Lambda,m)} \), then \((i, j)-\{x\}^{(\Lambda,m)} \subseteq (j, i)-\{y\}^{(\Lambda,m)}\), where \( i, j = 1, 2 \) and \( i \neq j \). And this contradicts the fact that \( z \not\in (j, i)-\{y\}^{(\Lambda,m)} \). Consequently, \( x \) belongs to the \((j, i)-(\Lambda, m)-open set\) \( X - (\{i, j\}-\{y\})^{(\Lambda, m)} \) to which \( y \) does not belong.

Conversely, let \((X, m^1_X, m^2_X)\) be a pairwise \( \Lambda_m-T_0 \) space and let \( x, y \) be any two distinct points of \( X \). There exists a \((i, j)-(\Lambda, m)-open set\) \( G \) containing \( x \) or \( y \), say \( x \) but not \( y \). Then \( X - G \) is a \((i, j)-(\Lambda, m)-closed set\) which does not contain \( x \) but contains \( y \). Since \((j, i)-\{y\}^{(\Lambda,m)} \) is the smallest \((j, i)-(\Lambda, m)-closed set\) containing \( y \) by Lemma 3.4, \((j, i)-\{y\}^{(\Lambda,m)} \subseteq X-G \), and so \( x \not\in (j, i)-\{y\}^{(\Lambda,m)} \). Consequently, \((i, j)-\{x\}^{(\Lambda,m)} \neq (j, i)-\{y\}^{(\Lambda,m)}\), where \( i, j = 1, 2 \) and \( i \neq j \). \( \square \)

Theorem 3.13. A biminimal structure space \( (X, m^1_X, m^2_X) \) is pairwise \( \Lambda_m-T_1 \) if and only if the singletons are \((i, j)-(\Lambda, m)-closed sets\), where \( i, j = 1, 2 \) and \( i \neq j \).

Proof. Suppose that \((X, m^1_X, m^2_X)\) is pairwise \( \Lambda_m-T_1 \) and \( x \) be any point of \( X \). Let \( y \in X - \{x\} \). Then \( x \neq y \) and there exists a \((i, j)-(\Lambda, m)-open set\) \( U_y \) such that \( y \in U_y \) but \( x \not\in U_y \). Consequently, \( y \in U_y \subseteq X - \{x\} \) i.e., \( X - \{x\} = \cup\{U_y : y \in X - \{x\}\} \) which is \((i, j)-(\Lambda, m)-open \).

Conversely, suppose that \( \{z\} \) is \((i, j)-(\Lambda, m)-closed \) for \( z \in X \), where \( i, j = 1, 2 \) and \( i \neq j \). Let \( x, y \in X \) with \( x \neq y \). Now \( x \neq y \) implies \( y \in X - \{x\} \). Hence, \( X - \{x\} \) is a \((i, j)-(\Lambda, m)-open set\) containing \( y \) but not containing \( x \). Similarly \( X - \{y\} \) is a \((j, i)-(\Lambda, m)-open set\) containing \( x \) but not containing \( y \). Therefore, \((X, m^1_X, m^2_X)\) is a pairwise \( \Lambda_m-T_1 \) space. \( \square \)

Definition 3.14. A biminimal structure space \( (X, m^1_X, m^2_X) \) is said to be a pairwise \( \Lambda_m\)-symmetric if, for \( x \) and \( y \) in \( X \), \( x \in (j, i)-\{y\}^{(\Lambda,m)} \) implies \( y \in (i, j)-\{x\}^{(\Lambda,m)} \), where \( i, j = 1, 2 \) and \( i \neq j \).

Definition 3.15. A subset \( A \) of a biminimal structure space \( (X, m^1_X, m^2_X) \) is called a \((i, j)-(\Lambda, m)\)-generalized closed set (briefly \((i, j)-(\Lambda_m)-g\)-closed) if \((j, i)-A^{(\Lambda,m)} \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \((i, j)-(\Lambda, m)\)-open, where \( i, j = 1, 2 \) and \( i \neq j \).

Lemma 3.16. Every \((i, j)-(\Lambda, m)\)-closed set is \((i, j)-(\Lambda_m)-g\)-closed, where \( i, j = 1, 2 \) and \( i \neq j \).

Remark 2. The converse of Lemma 3.16 is not true as shown in the following example.
Example 3.17. Let $X = \{a, b, c, d\}$, $m_X^1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ and $m_X^2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then $\{c\}$ is $(1, 2)$-$\Lambda_{m}$-g-closed but not $(1, 2)$-$\Lambda_{m}$-closed.

Theorem 3.18. A biminimal structure space $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_m$-symmetric if and only if $\{x\}$ is $(i, j)$-$\Lambda_{m}$-g-closed for each $x \in X$, where $i, j = 1, 2$ and $i \neq j$.

Proof. Assume that $x \in (j, i)-\{y\}^{(\Lambda, m)}$ but $y \in (i, j)-\{x\}^{(\Lambda, m)}$. This implies that the complement of $(i, j)-\{x\}^{(\Lambda, m)}$ contains $y$. Therefore, the set $\{y\}$ is a subset of the complement of $(i, j)-\{x\}^{(\Lambda, m)}$. This implies that $(j, i)-\{y\}^{(\Lambda, m)}$ is a subset of the complement of $(i, j)-\{x\}^{(\Lambda, m)}$. Now the complement of $(i, j)-\{x\}^{(\Lambda, m)}$ contain $x$ which is a contradiction.

Conversely, suppose that $\{x\} \subseteq G \in (i, j)-\Lambda_{m}O(X)$, but $(i, j)-\{x\}^{(\Lambda, m)}$ is not a subset of $G$. This mean that $(i, j)-\{x\}^{(\Lambda, m)}$ and the complement of $G$ are not disjoint. Let $y \in (i, j)-\{x\}^{(\Lambda, m)} \cap (X - G)$. Now we have $x \in (j, i)-\{y\}^{(\Lambda, m)}$ which is a subset of the complement of $G$ and $x \notin G$. But this is a contradiction. \hfill \Box

Corollary 3.19. If a biminimal structure space $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_{m}$-$T_1$ space, then it is pairwise $\Lambda_{m}$-symmetric.

Proof. In a pairwise $\Lambda_{m}$-$T_1$ space singleton sets are $(i, j)$-$\Lambda_{m}$-closed by Theorem 3.13, and therefore, $(i, j)$-$\Lambda_{m}$-g-closed by Lemma 3.16. By Theorem 3.18, the space is pairwise $\Lambda_{m}$-symmetric. \hfill \Box

Remark 3. The converse of Corollary 3.19 is not true as shown in the following example.

Example 3.20. Let $X = \{a, b, c\}$, $m_X^1 = \{\emptyset, \{a, b\}, X\}$ and $m_X^2 = \{\emptyset, \{b, c\}, X\}$. Then $(X, m_X^1, m_X^2)$ is a pairwise $\Lambda_{m}$-symmetric but not pairwise $\Lambda_{m}$-$T_1$ since $X$ and $\emptyset$ are the only $(i, j)$-$\Lambda_{m}$-open sets for $i, j = 1, 2$ and $i \neq j$.

Corollary 3.21. For a biminimal structure space $(X, m_X^1, m_X^2)$, the following are equivalent:

1. $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_{m}$-symmetric and pairwise $\Lambda_{m}$-$T_0$;

2. $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_{m}$-$T_1$.

Proof. By Corollary 3.19 and Remark 1, it suffices to prove only (1) $\Rightarrow$ (2). Let $x \neq y$ and by pairwise $\Lambda_{m}$-$T_0$, we may assume that $x \in G_1 \subseteq X - \{y\}$ for some $G_1 \in (i, j)-\Lambda_{m}O(X)$. Then $x \notin (j, i)-\{y\}^{(\Lambda, m)}$. Therefore, we have $y \notin (i, j)-\{x\}^{(\Lambda, m)}$. There exists a $G_2 \in (j, i)-\Lambda_{m}O(X)$ such that $y \in G_2 \subseteq X - \{x\}$.

Therefore, $(X, m_X^1, m_X^2)$ is a pairwise $\Lambda_{m}$-$T_1$ space. \hfill \Box
Theorem 3.22. For a pairwise $\Lambda_m$-symmetric space $(X, m_X^1, m_X^2)$, the following are equivalent:

1. $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_m$-$T_0$;
2. $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_m$-$T_1$.

Proof. (1) $\Rightarrow$ (2): Follows from Corollary 3.21. (2) $\Rightarrow$ (1): Follows from Remark 1.

4 Pairwise $\Lambda_m$-$R_0$ spaces

Definition 4.1. A biminimal structure space $(X, m_X^1, m_X^2)$ is said to be a pairwise $\Lambda_m$-$R_0$ if, for each $(i, j)$-$(\Lambda, m)$-open set $G$, $x \in G$ implies $(j, i)$-$\{x\}^{(\Lambda,m)} \subseteq G$, where $i, j = 1, 2$ and $i \neq j$.

Proposition 4.2. Let $(X, m_X^1, m_X^2)$ be a biminimal structure space, where $m_X^1$, $m_X^2$ have property $\mathcal{B}$. Then the following statements are equivalent:

1. $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_m$-$R_0$;
2. for any $(i, j)$-$(\Lambda, m)$-closed set $F$ and $x \notin F$, there exists $U \in (j, i)$-$\Lambda_mO(X)$ such that $x \notin U$ and $F \subseteq U$, where $i, j = 1, 2$ and $i \neq j$.
3. for any $(i, j)$-$(\Lambda, m)$-closed set $F$ and $x \notin F$, then $(j, i)$-$\{x\}^{(\Lambda,m)} \cap F = \emptyset$, where $i, j = 1, 2$ and $i \neq j$.

Proof. (1) $\Rightarrow$ (2): Let $F$ be a $(i, j)$-$(\Lambda, m)$-closed set and $x \notin F$. Then by (1), $(j, i)$-$\{x\}^{(\Lambda,m)} \subseteq X - F$, where $i, j = 1, 2$ and $i \neq j$. Let $U = X - ((j, i)$-$\{x\}^{(\Lambda,m)}))$, then $U \in (j, i)$-$\Lambda_mO(X)$ and also $F \subseteq U$ and $x \notin U$.

(2) $\Rightarrow$ (3): Let $F$ be a $(i, j)$-$(\Lambda, m)$-closed set and a point $x \notin F$. Then by (2), there exists $U \in (j, i)$-$\Lambda_mO(X)$ such that $F \subseteq U$ and $x \notin U$, where $i, j = 1, 2$ and $i \neq j$. Since $U \in (j, i)$-$\Lambda_mO(X)$, $U \cap (j, i)$-$\{x\}^{(\Lambda,m)} = \emptyset$. Then $F \cap (j, i)$-$\{x\}^{(\Lambda,m)} = \emptyset$, where $i, j = 1, 2$ and $i \neq j$.

(3) $\Rightarrow$ (1): Let $G \in (i, j)$-$\Lambda_mO(X)$ and $x \in G$. Now $X - G$ is $(i, j)$-$(\Lambda, m)$-closed and $x \notin X - G$. By (3), $(j, i)$-$\{x\}^{(\Lambda,m)} \cap (X - G) = \emptyset$ and hence $(j, i)$-$\{x\}^{(\Lambda,m)} \subseteq G$, where $i, j = 1, 2$ and $i \neq j$. Therefore, $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_m$-$R_0$.

Proposition 4.3. A biminimal structure space $(X, m_X^1, m_X^2)$ is pairwise $\Lambda_m$-$R_0$ if and only if, for each pair $x, y$ of distinct points in $X$, $(1, 2)$-$\{x\}^{(\Lambda,m)} \cap (2, 1)$-$\{y\}^{(\Lambda,m)} = \emptyset$ or $\{x, y\} \subseteq (1, 2)$-$\{x\}^{(\Lambda,m)} \cap (2, 1)$-$\{y\}^{(\Lambda,m)}$. 


Proof. Let \((X, m^1_X, m^2_X)\) be pairwise \(\Lambda_m\)-\(R_0\). Suppose that \((1, 2)\{-x\}^{(\Lambda, m)} \cap (2, 1)\{-y\}^{(\Lambda, m)} \neq \emptyset\) and \(\{x, y\}\) is not a subset of \((1, 2)\{-x\}^{(\Lambda, m)} \cap (2, 1)\{-y\}^{(\Lambda, m)}\). Let \(z \in (1, 2)\{-x\}^{(\Lambda, m)} \cap (2, 1)\{-y\}^{(\Lambda, m)}\) and \(x \not\in (1, 2)\{-x\}^{(\Lambda, m)} \cap (2, 1)\{-y\}^{(\Lambda, m)}\).

Then \(x \not\in (2, 1)\{-y\}^{(\Lambda, m)}\) and \(x \in X - ((2, 1)\{-y\}^{(\Lambda, m)}) \in (2, 1)\{-y\}^{(\Lambda, m)}\). But \((1, 2)\{-x\}^{(\Lambda, m)}\) is not a subset of \(X - ((2, 1)\{-y\}^{(\Lambda, m)})\) since this is a contradiction. Hence, for each point \(x, y\) of distinct points in \(X, (1, 2)\{-x\}^{(\Lambda, m)} \cap (2, 1)\{-y\}^{(\Lambda, m)}\).

Conversely, let \(U\) be a \((1, 2)\{-\Lambda, m\}\)-open set and \(x \in U\). Suppose that \((2, 1)\{-x\}^{(\Lambda, m)}\) is not a subset of \(U\). So there is a point \(y \in (2, 1)\{-x\}^{(\Lambda, m)}\) such that \(y \not\in U\) and \((1, 2)\{-y\}^{(\Lambda, m)} \cap U = \emptyset\). Since \(X - U\) is \((1, 2)\{-\Lambda, m\}\)-closed and \(y \in X - U\). Hence, \(\{x, y\}\) is not a subset of \((1, 2)\{-y\}^{(\Lambda, m)} \cap (2, 1)\{-x\}^{(\Lambda, m)}\) and thus \((1, 2)\{-y\}^{(\Lambda, m)} \cap (2, 1)\{-x\}^{(\Lambda, m)} \neq \emptyset\).

\[\square\]

Theorem 4.4. Let \((X, m^1_X, m^2_X)\) be a biminimal structure space, where \(m^1_X, m^2_X\) have property \(\mathcal{B}\). Then the following statements are equivalent:

1. \((X, m^1_X, m^2_X)\) is pairwise \(\Lambda_m\)-\(R_0\);
2. For any \(x \in X\), \((i, j)\{-x\}^{(\Lambda, m)} = (j, i)\{-x\}^{(\Lambda, m)}\), where \(i, j = 1, 2\) and \(i \neq j\);
3. For any \(x \in X\), \((i, j)\{-x\}^{(\Lambda, m)} \subseteq (j, i)\{-x\}^{(\Lambda, m)}\), where \(i, j = 1, 2\) and \(i \neq j\);
4. For any \(x, y \in X\), \(y \in (i, j)\{-x\}^{(\Lambda, m)}\) if and only if \(x \in (j, i)\{-y\}^{(\Lambda, m)}\), where \(i, j = 1, 2\) and \(i \neq j\);
5. For any \((i, j)\{-\Lambda, m\}\)-closed set \(F, F = \cap\{G : G \supseteq F, G \in (i, j)\{-\Lambda_m\}O(X)\}\), where \(i, j = 1, 2\) and \(i \neq j\);
6. For any \((i, j)\{-\Lambda, m\}\)-open set \(G, G = \cup\{F : F \subseteq G, F \in (i, j)\{-\Lambda_m\}C(X)\}\), where \(i, j = 1, 2\) and \(i \neq j\);
7. For any nonempty set \(A\) and each \((i, j)\{-\Lambda, m\}\)-open set \(G\) such that \(A \cap G \neq \emptyset\), there exists a \((j, i)\{-\Lambda, m\}\)-closed set \(F\) such that \(F \subseteq G\) and \(A \cap F \neq \emptyset\), where \(i, j = 1, 2\) and \(i \neq j\).

Proof. (1) \(\Rightarrow\) (2): Let \(x, y \in X\). Then by Lemma 3.8 and Proposition 4.3, \(y \in (j, i)\{-x\}^{(\Lambda, m)} \leftrightarrow x \in (i, j)\{-y\}^{(\Lambda, m)} \leftrightarrow y \in (j, i)\{-x\}^{(\Lambda, m)}\). Hence, \((i, j)\{-x\}^{(\Lambda, m)} = (j, i)\{-x\}^{(\Lambda, m)}\), where \(i, j = 1, 2\) and \(i \neq j\).

(2) \(\Rightarrow\) (3): This is obvious.

(3) \(\Rightarrow\) (4): For any \(x, y \in X\), if \(y \in (i, j)\{-x\}^{(\Lambda, m)}\), then \(y \in (j, i)\{-x\}^{(\Lambda, m)}\) by (3). Then by Lemma 3.8, \(x \in (j, i)\{-y\}^{(\Lambda, m)}\), where \(i, j = 1, 2\) and \(i \neq j\). The converse follows by the same token.

(4) \(\Rightarrow\) (5): Let \(F\) be a \((i, j)\{-\Lambda, m\}\)-closed set and \(H = \cap\{G : G \supseteq F, G \in (i, j)\{-\Lambda_m\}O(X)\}\). Clearly, \(F \subseteq H\). Let \(x \not\in F\). Then, for any \(y \in F\), we have
that \((i,j)\)-\{-\(y\}\}^{(\Lambda,m)} \subseteq F\). Hence, follows that \(x \notin (i,j)\)-\{-\(y\}\}^{(\Lambda,m)}\). Now by (4), \(x \notin (i,j)\)-\{-\(y\}\}^{(\Lambda,m)}\) implies \(y \notin (j,i)\)-\{-\(x\}\}^{(\Lambda,m)}\). There exists a \((j,i)\)-(\(\Lambda,m\))-open set \(G_y\) such that \(y \in G_y\) and \(x \notin G_y\). Let \(G = \bigcup_{y \in F} \{G_y : G_y \in (i,j)\)-\(\Lambda_mO(X), y \in G_y\text{ and } x \notin G_y\}\). Thus, there exists a \((j,i)\)-(\(\Lambda,m\))-open set \(G\) such that \(x \notin G\) and \(F \subseteq G\). Hence, \(x \notin H\). Consequently, \(F = H\).

(5) \(\Rightarrow\) (6): This is obvious.

(6) \(\Rightarrow\) (7): Let \(A\) be a nonempty set of \(X\) and \(G\) be a \((i,j)\)-(\(\Lambda,m\))-open set such that \(A \cap G \neq \emptyset\). There exists \(x \in A \cap G\). By (6), \(G = \bigcup \{F : F \subseteq G, F \in (i,j)\)-\(\Lambda_mC(X)\}\). It follows that there is a \((i,j)\)-(\(\Lambda,m\))-closed set \(F\) such that \(x \in F \subseteq G\). Hence, \(A \cap F \neq \emptyset\).

(7) \(\Rightarrow\) (1): Let \(G\) be a \((i,j)\)-(\(\Lambda,m\))-open set and \(x \in G\), then \(\{x\} \cap G \neq \emptyset\).

By (7), there exists a \((j,i)\)-(\(\Lambda,m\))-closed set \(F\) such that \(x \in F \subseteq G\) and \(\{x\} \cap F \neq \emptyset\), which implies \((j,i)\)-\{-\(x\}\}^{(\Lambda,m)} \subseteq G\), where \(i,j = 1,2\) and \(i \neq j\). Therefore, \((X,m_1^X,m_2^X)\) is pairwise \(\Lambda_m\)-\(R_0\).

**Remark 4.** Let \((X,m_1^X,m_2^X)\) be a biminimal structure space. Then, for each \(x \in X\), let \(\diamond \{-\(x\}\}^{(\Lambda,m)} = (1,2)-\{-\(x\}\}^{(\Lambda,m)} \cap (2,1)-\{-\(x\}\}^{(\Lambda,m)}\) and \(\diamond \{-\(y\}\}^{(\Lambda,m)} = (1,2)-\Lambda_m\text{Ker}(\{x\}) \cap (2,1)-\Lambda_m\text{Ker}(\{x\})\).

**Proposition 4.5.** If a biminimal structure space \((X,m_1^X,m_2^X)\) is pairwise \(\Lambda_m\)-\(R_0\), then for each pair of distinct points \(x,y \in X\), either \(\diamond \{-\(x\}\}^{(\Lambda,m)} \neq \diamond \{-\(y\}\}^{(\Lambda,m)}\) or \(\diamond \{-\(x\}\}^{(\Lambda,m)} \cap \diamond \{-\(y\}\}^{(\Lambda,m)} = \emptyset\).

**Proof.** Let \((X,m_1^X,m_2^X)\) be a pairwise \(\Lambda_m\)-\(R_0\) space. Suppose that \(\diamond \{-\(x\}\}^{(\Lambda,m)} \neq \diamond \{-\(y\}\}^{(\Lambda,m)}\) and \(\diamond \{-\(x\}\}^{(\Lambda,m)} \cap \diamond \{-\(y\}\}^{(\Lambda,m)} \neq \emptyset\). Let \(z \in \diamond \{-\(x\}\}^{(\Lambda,m)} \cap \diamond \{-\(y\}\}^{(\Lambda,m)}\) and \(x \notin \diamond \{-\(y\}\}^{(\Lambda,m)} = (1,2)-\{-\(x\}\}^{(\Lambda,m)} \cap (2,1)-\{-\(x\}\}^{(\Lambda,m)}\). Then \(x \notin (i,j)\)-(\(\Lambda,m\))-\(\Lambda_mO(X)\), where \(i,j = 1,2\) and \(i \neq j\). But \((j,i)\)-\{-\(y\}\}^{(\Lambda,m)}\) is not a subset of \(X - ((i,j)\)-\{-\(y\}\}^{(\Lambda,m)})\) since \(z \in \diamond \{-\(x\}\}^{(\Lambda,m)} \cap \diamond \{-\(y\}\}^{(\Lambda,m)}\). Thus \((X,m_1^X,m_2^X)\) is not a pairwise \(\Lambda_m\)-\(R_0\) space which is a contradiction to our assumption. Hence, we have either \(\diamond \{-\(x\}\}^{(\Lambda,m)} = \diamond \{-\(y\}\}^{(\Lambda,m)}\) or \(\diamond \{-\(x\}\}^{(\Lambda,m)} \cap \diamond \{-\(y\}\}^{(\Lambda,m)} = \emptyset\).

**Theorem 4.6.** For a biminimal structure space \((X,m_1^X,m_2^X)\), the following statements are equivalent:

(1) \((X,m_1^X,m_2^X)\) is pairwise \(\Lambda_m\)-\(R_0\);

(2) For any \((i,j)\)-(\(\Lambda,m\))-closed set \(F \subseteq X\), \(F = (j,i)\)-\(\Lambda_m\text{Ker}(F)\), where \(i,j = 1,2\) and \(i \neq j\);

(3) For any \((i,j)\)-(\(\Lambda,m\))-closed set \(F \subseteq X\) and \(x \in F\), \((j,i)\)-\(\Lambda_m\text{Ker}(\{x\}) \subseteq F\), where \(i,j = 1,2\) and \(i \neq j\);

(4) For any \(x \in X\), \((j,i)\)-\(\Lambda_m\text{Ker}(\{x\}) \subseteq (i,j)\)-\{-\(x\}\}^{(\Lambda,m)}\), where \(i,j = 1,2\) and \(i \neq j\);
Proof. (1) ⇒ (2): Let $F$ be a $(i, j)$-$\Lambda_{m}$-closed and $x \notin F$. Then $X - F$ is $(i, j)$-$\Lambda_{m}$-open containing $x$. Since $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{0}$, $(j, i)$-$\{x\}^{(\Lambda, m)} \subseteq X - F$, where $i$, $j = 1, 2$, and $i \neq j$. Therefore, $(j, i)$-$\{x\}^{(\Lambda, m)} \cap F = \emptyset$ and by Proposition 3.10, $x \notin (j, i)$-$\Lambda_{m}$Ker$(F)$. Hence, $F = (j, i)$-$\Lambda_{m}$Ker$(F)$, where $i, j = 1, 2$, and $i \neq j$.

(2) ⇒ (3): Let $F$ be a $(i, j)$-$\Lambda_{m}$-closed set containing $x$. Then $\{x\} \subseteq F$ and $(j, i)$-$\Lambda_{m}$Ker$(\{x\}) \subseteq (j, i)$-$\Lambda_{m}$Ker$(F)$. From (2), it follows that $(j, i)$-$\Lambda_{m}$Ker$(\{x\}) \subseteq F$, where $i, j = 1, 2$, and $i \neq j$.

(3) ⇒ (4): Since $x \in (i, j)$-$\{x\}^{(\Lambda, m)}$ and $(i, j)$-$\{x\}^{(\Lambda, m)}$ is $(i, j)$-$\Lambda_{m}$-closed in $X$, by (3) it follows that $(j, i)$-$\Lambda_{m}$Ker$(F) \subseteq (i, j)$-$\{x\}^{(\Lambda, m)}$, where $i, j = 1, 2$, and $i \neq j$.

(4) ⇒ (1): It follows from Theorem 4.4. \hfill \Box

5 Pairwise $\Lambda_{m}$-$R_{1}$ spaces

Definition 5.1. A biminimal structure space $(X, m_{X}^{1}, m_{X}^{2})$ is said to be a pairwise $\Lambda_{m}$-$R_{1}$ if, for each $x, y \in X$, $(i, j)$-$\{x\}^{(\Lambda, m)} \neq (j, i)$-$\{y\}^{(\Lambda, m)}$, there exist disjoint sets $U \in (j, i)$-$\Lambda_{m}O(X)$ and $V \in (i, j)$-$\Lambda_{m}O(X)$ such that $(i, j)$-$\{x\}^{(\Lambda, m)} \subseteq U$ and $(j, i)$-$\{y\}^{(\Lambda, m)} \subseteq V$, where $i, j = 1, 2$, and $i \neq j$.

Proposition 5.2. Let $(X, m_{X}^{1}, m_{X}^{2})$ be a biminimal structure space, where $m_{X}^{1}$, $m_{X}^{2}$ have property $B$. If $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{1}$, then it is pairwise $\Lambda_{m}$-$R_{0}$.

Proof. Suppose that $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{1}$. Let $U$ be a $(i, j)$-$\Lambda_{m}$-open set and $x \in U$. Then, for each point $y \in X - U$, $(j, i)$-$\{x\}^{(\Lambda, m)} \neq (i, j)$-$\{y\}^{(\Lambda, m)}$. Since $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{1}$, there exists a $(i, j)$-$\Lambda_{m}$-open set $U_{y}$ and a $(j, i)$-$\Lambda_{m}$-open set $V_{y}$ such that $(j, i)$-$\{x\}^{(\Lambda, m)} \subseteq U_{y}$ and $(i, j)$-$\{y\}^{(\Lambda, m)} \subseteq V_{y}$ and $U_{y} \cap V_{y} = \emptyset$, where $i, j = 1, 2$, and $i \neq j$. Let $G = \cup\{V_{y} : y \in X - U\}$. Then $X - U \subseteq G$, $x \notin G$, and $G$ is a $(j, i)$-$\Lambda_{m}$-open set. Therefore, $(j, i)$-$\{x\}^{(\Lambda, m)} \subseteq X - G \subseteq U$. Hence, $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{0}$. \hfill \Box

Proposition 5.3. A biminimal structure space $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{1}$ if and only if, for every pair of points $x$ and $y$ of $X$ such that $(i, j)$-$\{x\}^{(\Lambda, m)} \neq (j, i)$-$\{y\}^{(\Lambda, m)}$, there exists a $(i, j)$-$\Lambda_{m}$-open set $U$ and $(j, i)$-$\Lambda_{m}$-open set $V$ such that $x \in V$, $y \in U$, and $U \cap V = \emptyset$, where $i, j = 1, 2$, and $i \neq j$.

Proof. Suppose that $(X, m_{X}^{1}, m_{X}^{2})$ is pairwise $\Lambda_{m}$-$R_{1}$. Let $x, y$ be points of $X$ such that $(i, j)$-$\{x\}^{(\Lambda, m)} \neq (j, i)$-$\{y\}^{(\Lambda, m)}$, where $i, j = 1, 2$, and $i \neq j$. Then, there exists a $(i, j)$-$\Lambda_{m}$-open set $U$ and $(j, i)$-$\Lambda_{m}$-open set $V$ such that $x \in (i, j)$-$\{x\}^{(\Lambda, m)} \subseteq V$ and $y \in (j, i)$-$\{y\}^{(\Lambda, m)} \subseteq U$. On the other hand, suppose
that there exists a \((i, j)-(\Lambda, m)\)-open set \(U\) and \((j, i)-(\Lambda, m)\)-open set \(V\) such that \(x \in V\), \(y \in U\) and \(U \cap V = \emptyset\), where \(i, j = 1, 2\) and \(i \neq j\). Since every pairwise \(\Lambda_m\)-space \(\Lambda_m-R_1\) is pairwise \(\Lambda_m-R_0\), \((i, j)\)-\(\{x\}^{(\Lambda, m)} \subseteq V\) and \((j, i)\)-\(\{y\}^{(\Lambda, m)} \subseteq U\). Hence, the claim. \( \square \)

**Proposition 5.4.** A pairwise \(\Lambda_m-R_0\) space \((X, m_X^1, m_X^2)\) is pairwise \(\Lambda_m-R_1\) if, for each pair of points \(x\) and \(y\) of \(X\) such that \((i, j)\)-\(\{x\}^{(\Lambda, m)} \cap \{y\}^{(\Lambda, m)} = \emptyset\), there exist disjoint sets \((i, j)-(\Lambda, m)\)-open \(U\) and \((j, i)-(\Lambda, m)\)-open \(V\) such that \(x \in U\), \(y \in V\), where \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** It follows directly from Definition 4.1 and Proposition 4.5. \( \square \)

**Theorem 5.5.** For a biminimal structure space \((X, m_X^1, m_X^2)\), the following statements are equivalent:

1. \((X, m_X^1, m_X^2)\) is pairwise \(\Lambda_m-R_1\);
2. For any two distinct points \(x, y \in X\), \((i, j)\)-\(\{x\}^{(\Lambda, m)} \neq (j, i)\)-\(\{y\}^{(\Lambda, m)}\) implies that there exists a \((i, j)-(\Lambda, m)\)-closed set \(F\) and a \((j, i)-(\Lambda, m)\)-closed set \(H\) such that \(x \in F\), \(y \in H\), \(x \notin H\), \(y \notin F\) and \(X = F \cup H\), where \(i, j = 1, 2\) and \(i \neq j\).

**Proof.** (1) \(\Rightarrow\) (2): Suppose that \((X, m_X^1, m_X^2)\) is pairwise \(\Lambda_m-R_1\). Let \(x, y \in X\) such that \((i, j)\)-\(\{x\}^{(\Lambda, m)} \neq (j, i)\)-\(\{y\}^{(\Lambda, m)}\). By Proposition 5.3, there exist disjoint sets \(V \in (i, j)-(\Lambda_m)O(X)\) and \(U \in (j, i)-(\Lambda_m)O(X)\) such that \(x \in U\) and \(y \in V\), where \(i, j = 1, 2\) and \(i \neq j\). Then \(F = X - V\) is a \((i, j)-(\Lambda, m)\)-closed set and \(H = X - U\) is a \((j, i)-(\Lambda, m)\)-closed set such that \(x \in F\), \(y \in H\), \(x \notin H\), \(y \notin F\) and \(X = F \cup H\), where \(i, j = 1, 2\) and \(i \neq j\).

(2) \(\Rightarrow\) (1): Let \(x, y \in X\) such that \((i, j)\)-\(\{x\}^{(\Lambda, m)} \neq (j, i)\)-\(\{y\}^{(\Lambda, m)}\), where \(i, j = 1, 2\) and \(i \neq j\). Hence, for any two distinct points \(x, y\) of \(X\), \((i, j)\)-\(\{x\}^{(\Lambda, m)} \cap (j, i)\)-\(\{y\}^{(\Lambda, m)} = \emptyset\), where \(i, j = 1, 2\) and \(i \neq j\). Then by Proposition 4.3, \((X, m_X^1, m_X^2)\) is pairwise \(\Lambda_m-R_0\). By (2), there exists a \((i, j)-(\Lambda, m)\)-closed set \(F\) and a \((j, i)-(\Lambda, m)\)-closed set \(H\) such that \(X = F \cup H\), \(x \in F\), \(y \in H\), \(x \notin H\), \(y \notin F\). Therefore, \(x \in X - H = U \in (j, i)-(\Lambda_m)O(X)\) and \(y \in X - F = V \in (i, j)-(\Lambda_m)O(X)\) which implies that \((i, j)\)-\(\{x\}^{(\Lambda, m)} \subseteq U\), \((j, i)\)-\(\{y\}^{(\Lambda, m)} \subseteq V\) and \(U \cap V = \emptyset\), where \(i, j = 1, 2\) and \(i \neq j\). Hence, \((X, m_X^1, m_X^2)\) is pairwise \(\Lambda_m-R_0\). \( \square \)

**Acknowledgement**

The authors would like to thank the Faculty of Science, Mahasarakham University for financial support.
References


Received: September, 2010