Uniformity of Double Designs

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Abstract
Doubling is a simple and powerful method to construct two-level fractional factorial designs of resolution IV. The objective of this paper is to discuss the issue of double designs in terms of uniformity measured by centered $L_2$-discrepancy. The sufficient and necessary condition of double design to be a uniform design is provided. In addition, a lower bound of centered $L_2$-discrepancy of double design is obtained, which can be used to assess the uniformity of double design.

Mathematics Subject Classification: 62K15, 62K10

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1 Introduction
Recently, there has been considerable interest in studying the issue of double design. Suppose $X$ is an $n \times k$ matrix with two distinct entries, +1 and -1. We call the $2n \times 2k$ matrix $\begin{pmatrix} X & X \\ X & -X \end{pmatrix}$ is the double of $X$, denoted by $D(X)$. Suppose $X$ defines an $n$-run design for $k$ two-level factors, where the two levels are denoted by +1 and -1, each column of $X$ corresponds to a factor and each row defines a factor-level combination. Then $D(X)$ defines a design which doubles both the run size and the number of factors of $X$. We call $D(X)$ is a double design of $X$, and $X$ is the original design of $D(X)$.

Doubling method was firstly used by Plackett and Burman [8] to construct orthogonal main-effect plans. Recently, Chen and Cheng [1] studied to construct a doubling design $D(X)$ of resolution IV via a regular fractional factorial design $X$ of resolution IV, and proved that there exists a projection design of $D(X)$ with resolution IV or higher. Xu and Cheng [9] developed a general complementary design theory for doubling designs. We say that a regular design of
resolution IV or higher is maximal if its resolution reduces to three whenever an extra factor is added. The authors in [1] and [9] respectively discuss the function of doubling method in constructing maximal designs with two-level. Ou and Qin [6] presented some links between the double design $D(X)$ and the original design $X$ via the tool of indicator function. The objective of this paper is to discuss the issue of double designs in terms of uniformity.

In this paper, we shall employ the centered $L_2$-discrepancy ($CD$ for short) as the measure of uniformity, see [3] and [4] for details. If one fractional factorial design has smaller centered $L_2$-discrepancy value, then this design possesses better uniformity. The uniformity criterion favors designs with the best uniformity. Ma, Fang and Lin [5] showed some relationships between uniformity and orthogonality based on the quadratic form of centered $L_2$-discrepancy. Fang and Qin [2] provided some methods to construct nearly uniform designs with large number of runs. Following this line, Ou, Chatterjee and Qin [7] obtained a lower bound of centered $L_2$-discrepancy of combined design for general two-level designs. In the spirit of the above authors, we shall express the square $CD$ value of the double design $D(X)$ as a quadratic form via the Kronecker calculus for factorial arrangements. Based on the quadratic form, we will provide the sufficient and necessary condition of the double design $D(X)$ to be a uniform design. In addition, we will obtain a lower bound of centered $L_2$-discrepancy of double design $D(X)$. Numerical results show that the lower bound is tight, which can be used to assess the uniformity of double design.

2 Preliminaries

Consider a two-level fractional factorial design $X$ with $n$ runs and $k$ factors whose high level is denoted by $+1$ and the low level is denoted by $-1$, respectively. The set of all two-level fractional factorial designs with $n$ runs and $k$ factors is denoted by $\mathcal{D}(n;2^k)$. An arbitrary two-level factorial design $X$ can be represented by an $n \times k$ matrix $T_X = (t_1, t_2, \ldots, t_k) = (t_{ij})$, where $t_{ij} = +1$ or $-1$. Each row of $T_X$ corresponds to a run in $X$ and each column to an experimental factor.

For a design $X \in \mathcal{D}(n;2^k)$, its centered $L_2$-discrepancy value, denoted by $CD(X)$, can be expressed in the following close form:

$$[CD(X)]^2 = \left(\frac{13}{12}\right)^k - \frac{2}{n} \sum_{i=1}^{n} \prod_{l=1}^{k} \left(1 + \frac{1}{2}|x_{il} - \frac{1}{2}| - \frac{1}{2}|x_{il} - 1|\right)$$

$$+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{k} \left(1 + \frac{1}{2}|x_{il} - \frac{1}{2}| + \frac{1}{2}|x_{jl} - \frac{1}{2}| - \frac{1}{2}|x_{il} - x_{jl}|\right)$$
\[
\left( \frac{13}{12} \right)^k - 2 \left( \frac{35}{32} \right)^k + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{k} \left( \frac{5}{4} - \frac{1}{2} |x_{il} - x_{jl}| \right),
\]

(1)

where \( x_{il} = (t_{il} + 2)/4 \), \( t_{il} \in T_X \).

Following (1), the centered \( L_2 \)-discrepancy value of \( D(X) \), denoted by \( CD(D(X)) \), can calculate by the following formula:

\[
[CD(D(X))]^2 = \left( \frac{13}{12} \right)^{2k} - 2 \left( \frac{35}{32} \right)^{2k} + \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{k} \left( \frac{5}{4} - \frac{1}{2} |x_{il} - x_{jl}| \right)^2
\]

\[
+ \frac{1}{2n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{l=1}^{k} \left( \frac{5}{4} - \frac{1}{2} |x_{il} - x_{jl}| \right) \left( \frac{5}{4} - \frac{1}{2} |x_{il} + x_{jl} - 1| \right).
\]

(2)

Let \( V \) be the set of all \( v(= 2^k) \) treatment combinations written in the lexicographic order. For any \( u \in V \) and \( X \in D(n; 2^k) \), let \( n_d(u) \) be the number of times the treatment combination \( u \) occurs in \( X \) and \( n_d \) be the \( v \times 1 \) vector with elements \( n_d(u) \) arranged in the lexicographic order. Define

\[
M = \begin{pmatrix} 25/16 & 1 \\ 1 & 25/16 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 5/4 & 5/4 \\ 5/4 & 5/4 \end{pmatrix},
\]

and, let

\[
M_k = \bigotimes_{i=1}^{k} M \quad \text{and} \quad Q_k = \bigotimes_{i=1}^{k} Q.
\]

Here the symbol \( \bigotimes \) represents the Kronecker product. Following the line of the authors in \([2, 5, 7]\), we can express \([CD(D(X))]^2 \) in formula (2) as a quadratic form of \( n_d \) as follows

\[
[CD(D(X))]^2 = \left( \frac{13}{12} \right)^{2k} - 2 \left( \frac{35}{32} \right)^{2k} + \frac{1}{2n^2} n_d'M_kn_d + \frac{1}{2n^2} n_d'Q_kn_d.
\]

(3)

### 3 Main Results

Let \( 1_s \) and \( I_s \) be respectively the \( s \times 1 \) vector with all elements unity and the \( s \times s \) identity matrix. Define

\[
L(0) = 1_2', \quad L(1) = I_2 \quad \text{and} \quad J_2 = 1_21_2'.
\]

(4)

Let \( \Omega \) be the set of all binary \( k \) tuples and, for \( 0 \leq j \leq k \), let \( \Omega_j \) be the set of all binary \( k \) tuples that has exactly \( j \) ones. For any \( x = (x_1, x_2, \ldots, x_k) \in \Omega \), define the matrix

\[
G(x) = \bigotimes_{i=1}^{k} L(x_i).
\]

(5)
It is to be noted that $G(x)$ is of order $2\sum x^j \times v$. Here the symbol $\otimes$ represents the Kronecker product.

Now from above, $M$ and $Q$ can be expressed as

$$M = L(0)'L(0) + \frac{9}{16} L(1)'L(1) \quad \text{and} \quad Q = \frac{5}{4} L(0)'L(0),$$  \hspace{1cm} (6)

and

$$Q_k = \bigotimes_{i=1}^{k} Q_k = \bigotimes_{i=1}^{k} \left[ \frac{5}{4} L(0)'L(0) + 0 L(1)'L(1) \right]$$

$$= \sum_{j=0}^{k} \left( \frac{5}{4} \right)^{k-j} 0^j \sum_{x \in \Omega_j} G(x)'G(x)$$

$$= \left( \frac{5}{4} \right)^k.$$

Hence one can write

$$[CD(D(X))]^2 = \left( \frac{13}{12} \right)^{2k} - 2 \left( \frac{35}{32} \right)^{2k} + \frac{1}{2} \left( \frac{5}{4} \right)^k + \frac{1}{2n^2} n_d'M_k n_d,$$  \hspace{1cm} (7)

which can sever as the uniformity criterion for finding optimal design over $D(n; 2^k)$.

The following theorem provides the sufficient and necessary condition of the double design $D(X)$ to be a uniform design.

**Theorem 3.1** $X$ minimizes $[CD(D(X))]^2$ over $D(n; 2^k)$ if and only if $n_d = (\frac{n}{2})^{1/2k}$.

**Proof.** From (7), and the Lagrange multiplier method, let

$$L(n_d, \lambda) = n_d'M_k n_d - 2\lambda(n_d'1_{2k} - n).$$

The following system of equations

$$\frac{\partial L(n_d, \lambda)}{\partial \lambda} = n_d'1_{2k} - n = 0$$
$$\frac{\partial L(n_d, \lambda)}{\partial n_d} = 2M_k n_d - 2\lambda 1_{2k} = 0$$

gives

$$\lambda = \frac{n}{1_{2k} M_k^{-1} 1_{2k}}, \quad p_d = \frac{n}{1_{2k} M_k^{-1} 1_{2k}} M_k^{-1} 1_{2k}.$$  

Noting that

$$M_k = \bigotimes_{r=1}^{k} M, \quad M^{-1} = \frac{16}{9} \left( I_2 - \frac{16}{41} J_2 \right),$$
it follows that
\[ M_k^{-1}1_{2^k} = \bigotimes_{r=1}^{s} (M^{-1}1_2) = \left(\frac{16}{11}\right)^k 1_{2^k}, \quad 1'_{2^k} M_k^{-1}1_{2^k} = \left(\frac{16}{11}\right)^k 2^k, \]
and then \( n_d = (\frac{n}{2^k})1_{2^k} \), hence the proof is completed.

Noting the sufficient and necessary condition of the double design \( D(X) \) to be a uniform design is identical to the sufficient and necessary condition of the original design \( X \) to be a uniform design, therefore, the following result is obviously.

**Corollary 3.2** Double design \( D(X) \) is an uniform design in \( D(2n;2^k) \) if and only if the original design \( X \) is an uniform design in \( D(n;2^k) \).

For \( 0 \leq j \leq k \), let \( w_j \) be the largest integer contained in \( n/2^j \), define
\[ \theta_j = n - 2^j w_j, \quad \theta^*_j = n w_j + \theta_j (1 + w_j), \]
the following theorem provides the lower bound of \( CD(D(X)) \).

**Theorem 3.3** For any \( X \in D(n;2^k) \), we have
\[ [CD(D(X))]^2 \geq LCD(n,k), \]
where
\[ LCD(n,k) = \left(\frac{13}{12}\right)^{2k} - 2\left(\frac{35}{32}\right)^{2k} + \frac{1}{2}\left(1 + \left(\frac{5}{4}\right)^k\right) + \frac{1}{2n^2} \sum_{j=1}^{k} \left(\frac{9}{16}\right)^j \binom{k}{j} \theta^*_j \]
and \( \binom{x}{y} = x(x-1)\cdots(x-y+1)/y! \), with \( \binom{x}{0} = 1 \) and \( \binom{x}{y} = 0 \) if \( x < y \).

**Proof.** From (5) and (6), we can express the matrix \( M_k \) as
\[ M_k = \bigotimes_{i=1}^{k} [L(0)'L(0) + \frac{9}{16}L(1)'L(1)] \]
\[ = \sum_{j=0}^{k} \sum_{x \in \Omega_j} \left(\frac{9}{16}\right)^j G(x)'G(x). \]
Therefore,
\[ n''_d M_k n_d = \sum_{j=0}^{k} \left(\frac{9}{16}\right)^j \sum_{x \in \Omega_j} n''_d G(x)'G(x)n_d. \]
Since \( X \) consists of \( n \) runs, from (4) and (5), we get
\[ \sum_{x \in \Omega_0} n''_d G(x)'G(x)n_d = n^2. \]
Now, from (7) and (9), we obtain

\[
[CD(D(X))]^2 = \left( \frac{13}{12} \right)^{2k} - 2 \left( \frac{35}{32} \right)^{2k} + \frac{1}{2} \left( 1 + \left( \frac{5}{4} \right)^k \right) \\
+ \frac{1}{2n^2} \sum_{j=1}^{k} \left( \frac{9}{16} \right)^j \sum_{x \in \Omega_j} n_d'G(x)^tG(x)n_d,
\tag{10}
\]

It is to be noted that the elements of the \(2^i \times 1\) vector \(G(x)n_d\) are nonnegative integers with \(1'_2 G(x)n_d = n\). Hence, for each \(x \in \Omega_j\),

\[
n_d'G(x)^tG(x)n_d \geq \theta^*_j.
\tag{11}
\]

From (10) and (11), the proof of (8) follows immediately.

Now we present an example to illustrate our theoretical results. It is noted that the lower bound given above is tight.

**Example 3.4** Consider the original design \(X \in D(4; 2^3)\) given in Table 1. From (2), we have \([CD(D(X))]^2 = 0.2318\), and from (8) we have \(LCD(4, 3) = 0.2318\).

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0.2318.

Noting that these results provided in this paper are suitable to general original two-level factorial design, whether regular or nonregular. These results provide strong insight of the relationship between double design and original design from uniform viewpoint. Furthermore, based on Corollary 3.2, one can easily construct a kind of uniform design \(X\) with large run size over \(D(2^i n; 2^{2i} k)\) via successively doubling, where design \(X\) has \(2^i n\) runs and \(2^i k\) factors and \(i\) is a positive integer.

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**References**

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