Hybrid Descent-Like Halpern Iteration Methods for Two Nonexpansive Mappings and Semigroups on Two Sets

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Abstract

In this paper, we introduce some new iteration methods based on the hybrid method in mathematical programming, the Mann’s iterative method and the Halpern’s method for finding a fixed point of a nonexpansive mapping and a common fixed point of a nonexpansive semigroup Hilbert spaces.

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1. Introduction

Let \( H \) be a real Hilbert space with the scalar product and the norm denoted by the symbols \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \), respectively, and let \( C \) be a nonempty closed and convex subset of \( H \). Denote by \( P_C(x) \) the metric projection from \( x \in H \) onto \( C \). Let \( T \) be a nonexpansive mapping on \( C \), i.e., \( T : C \to C \) and
\[\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.\] We use \(F(T)\) to denote the set of fixed points of \(T\), i.e., \(F(T) = \{x \in C : x = Tx\}\). We know that \(F(T)\) is nonempty, if \(C\) is bounded, for more details see [1].

Let \(\{T(t) : t > 0\}\) be a nonexpansive semigroup on \(C\), that is,

1. for each \(t > 0\), \(T(t)\) is a nonexpansive mapping on \(C\);
2. \(T(0)x = x\) for all \(x \in C\);
3. \(T(t_1 + t_2) = T(t_1) \circ T(t_2)\) for all \(t_1, t_2 > 0\); and
4. for each \(x \in C\), the mapping \(T(\cdot)x\) from \((0, \infty)\) into \(C\) is continuous.

Denote by \(\mathcal{F} = \cap_{t > 0} F(T(t))\) the set of common fixed points for the semigroup \(\{T(t) : t > 0\}\). We know that \(\mathcal{F}\) is a closed convex subset in \(H\) and \(\mathcal{F} \neq \emptyset\) if \(C\) is compact (see, [2]).

For finding a fixed point of a nonexpansive mapping \(T\) on \(C\), in 1953, Mann [3] proposed the following method:

\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Tx_n, \quad (1.1)
\]

that converges only weakly, in general (see [4] for an example). In 1967, Halpern [5] firstly proposed the following iteration process:

\[
x_{n+1} = \beta_n u + (1 - \beta_n) Tx_n, \quad n \geq 0, \quad (1.2)
\]

where \(u, x_0\) are two fixed elements in \(C\) and \(\{\beta_n\} \subset (0, 1)\). He pointed out that the conditions \(\lim_{n \to \infty} \beta_n = 0\) and \(\sum_{n=0}^{\infty} \beta_n = \infty\) are necessary in the sense that, if the iteration (1.2) converges to a fixed point of \(T\), then these conditions must be satisfied. Further, the iteration method was investigated by Lions [6], Reich [7], Wittmann [8] and Song [9]. Recently, Alber [10] proposed the following descent-like method

\[
x_{n+1} = P_C(x_n - \mu_n [x_n - Tx_n]), \quad n \geq 0, \quad (1.3)
\]

and proved that if \(\{\mu_n\} : \mu_n > 0, \mu_n \to 0\), as \(n \to \infty\) and \(\{x_n\}\) is bounded, then:

(i) there exists a weak accumulation point \(\bar{x} \in C\) of \(\{x_n\}\);
(ii) all weak accumulation points of \(\{x_n\}\) belong to \(F(T)\); and
(iii) if \(F(T)\) is a singleton, i.e., \(F(T) = \{\bar{x}\}\), then \(\{x_n\}\) converges weakly to \(\bar{x}\).
To obtain strong convergence for (1.1), Nakajo and Takahashi [11] introduced the hybrid Mann’s iteration method:

$$x_0 \in C,$$
$$y_n = \alpha_n x_n + (1 - \alpha_n) T x_n,$$
$$C_n = \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \},$$
$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \},$$
$$x_{n+1} = P_{C_n \cap Q_n}(x_0),$$

(1.4)

where \( \{ \alpha_n \} \subset [0,a] \) for some \( a \in [0,1) \). They showed that \( \{ x_n \} \) defined by (1.4) converges strongly to \( P_{F(T)}(x_0) \) as \( n \to \infty \). Recently, Yanes and Xu [12] adapted the iteration process (1.2) as follows:

$$x_0 \in C \text{ any element,}$$
$$y_n = \beta_n x_0 + (1 - \beta_n) T x_n,$$
$$C_n = \{ z \in C : \| y_n - z \|^2 \leq \| x_n - z \|^2$$
$$+ \beta_n(\| x_0 \|^2 + 2 \langle x_n - x_0, z \rangle) \},$$
$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \},$$
$$x_{n+1} = P_{C_n \cap Q_n}(x_0),$$

(1.5)

They proved that if \( T \) is a nonexpansive mapping on a closed convex subset \( C \) with \( F(T) \neq \emptyset \) and the sequence \( \{ \beta_n \} \subset (0,1) \) is chosen such that \( \lim_{n \to \infty} \beta_n = 0 \), then the sequence \( \{ x_n \} \) defined by (1.5) converges strongly to \( P_{F(T)}(x_0) \) as \( n \to \infty \).

For finding an element \( p \in F \), Nakajo and Takahashi [11] also introduced an iteration procedure as follows:

$$x_0 \in C \text{ any element,}$$
$$y_n = \alpha_n x_n + (1 - \alpha_n) \frac{1}{t_n} \int_{0}^{t_n} T(s)x_n ds,$$
$$C_n = \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \},$$
$$Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \},$$
$$x_{n+1} = P_{C_n \cap Q_n}(x_0), n \geq 0,$$

(1.6)

where \( \alpha_n \in [0,a] \) for some \( a \in [0,1) \) and \( \{ t_n \} \) is a positive real number divergent sequence. Under the conditions on \( \{ \alpha_n \} \) and \( \{ t_n \} \), the sequence \( \{ x_n \} \) defined by (1.6) converges strongly to \( P_F(x_0) \).
If $C \equiv H$, then $C_n$ and $Q_n$ in (1.4)-(1.6) are two halfspaces. So, the projection $x_{n+1}$ onto $C_n \cap Q_n$ in these methods can be described by an explicit formula [13]. Clearly, if $C$ is a proper subset of $H$, then $C_n$ and $Q_n$ in (1.4)-(1.6) are not two halfspaces. Then, the following big problem is posed: how to construct the closed convex subsets $C_n$ and $Q_n$ in (1.4)-(1.6) in a similar form as in [13]? This problem is solved very recently in [14] and [15]. In this works, $C_n$ and $Q_n$ are replaced by two halfspaces and $y_n$ is the right hand side of (1.3) with a modification.

In this paper, using the idea, we introduce the following new iteration processes:

$x_0 \in H$ any element,

$z_n = \alpha_n P_C(x_n) + (1 - \alpha_n) P_C TP_C(x_n)$,

$y_n = \beta_n x_0 + (1 - \beta_n) P_C T z_n$,

$H_n = \{ z \in H : \| y_n - z \|^2 \leq \| x_n - z \|^2$

$+ \beta_n (\| x_0 \|^2 + 2 \langle x_n - x_0, z \rangle) \}$,

$W_n = \{ z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}$,

$x_{n+1} = P_{H_n \cap W_n}(x_0), n \geq 0$;

and

$x_0 \in H$ any element,

$z_n = \alpha_n P_C(x_n) + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) P_C(x_n) ds$,

$y_n = \beta_n x_0 + (1 - \beta_n) \frac{1}{t_n} \int_0^{t_n} T(s) z_n ds$,

$H_n = \{ z \in H : \| y_n - z \|^2 \leq \| x_n - z \|^2$

$+ \beta_n (\| x_0 \|^2 + 2 \langle x_n - x_0, z \rangle) \}$,

$W_n = \{ z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}$,

$x_{n+1} = P_{H_n \cap W_n}(x_0), n \geq 0$;

for a nonexpansive mapping $T : C \rightarrow H$ and a nonexpansive semigroup $\{ T(t) : t > 0 \}$ on $C$, respectively. We shall prove the strong convergence of the sequences $\{ x_n \}, \{ y_n \}$ and $\{ z_n \}$ defined by (1.7) and (1.8) to a fixed point of $T$ and a common fixed point of the nonexpansive semigroup $\{ T(t) : t > 0 \}$, respectively.

Later, the symbols $\rightharpoonup$ and $\rightarrow$ denote weak and strong convergences, respectively.

2. Strong convergence to a fixed point of nonexpansive mappings
We formulate the following facts needed in the proof of our results.

**Lemma 2.1** [16]. Let $H$ be a real Hilbert space $H$. There holds the following identity: $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$.

**Lemma 2.2** [12]. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For any $x \in H$, there exists a unique $z \in C$ such that $\|z - x\| \leq \|y - x\|$ for all $y \in C$, and $z = P_C(x)$ if and only if $\langle z - x, y - z \rangle \geq 0$ for all $y \in C$, where $P_C$ is the metric projection of $H$ on $C$.

**Lemma 2.3.** (Demiclosedness principle) [17]. If $C$ is a nonempty closed convex subset of a real Hilbert space $H$, $T$ is a nonexpansive mapping on $C$, $\{x_n\}$ is a sequence in $C$ such that $x_n \to x$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.

**Lemma 2.4** [17]. Every Hilbert space $H$ has Randon-Riesz property or Kadec-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \to x$ and $\|x_n\| \to \|x\|$, then there holds $x_n \to x$.

Now, we are in a position to prove the following result.

**Theorem 2.5.** Let $C$ be a nonempty closed convex subset in a real Hilbert space $H$ and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\alpha_n \to 1$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.7) converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.

**Proof.** First, note that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z \rangle)$$

is equivalent to

$$(1 - \beta_n)\|y_n - x_n\|^2 - \beta_n\|y_n - x_n\|^2 \leq \|x_n - y_n, x_n\| - \frac{1}{2}\|y_n - x_n\|^2 + \frac{\beta_n}{2}\|x_0\|^2.$$

Thus, $H_n$ is a halfspace. It is clear that $F(T_1) = F(T_1P_C) := \{p \in H : T_1P_C(p) = p\}$ for any mapping $T_1$ from $C$ into $C$. Taking $T_1 = P_CT$ and using Lemma 2.6 in [15] with $S = P_CT$, we have that $F(T) = F(P_CTPC)$. Hence, by the convexity of $\|\cdot\|^2$ and the nonexpansive property of $P_C$, we obtain for any $p \in F(T)$ that $p = P_CTPC(p)$, and hence

$$\|z_n - p\|^2 = \|\alpha_nP_C(x_n) - p + (1 - \alpha_n)P_CTPC(x_n)\|^2$$

$$= \|\alpha_n(P_C(x_n) - P_C(p)) + (1 - \alpha_n)[P_CTPC(x_n) - P_CTPC(p)]\|^2$$

$$\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|P_C(x_n) - P_C(p)\|^2$$

$$\leq \|x_n - p\|^2.$$
By the similar argument and Lemma 2.1 with \( x = x_0 - p \) and \( y = x_n - p \), we also obtain

\[
\|y_n - p\|^2 = \|\beta_n x_0 + (1 - \beta_n)P_C T z_n - p\|^2 \\
\leq \beta_n \|x_0 - p\|^2 + (1 - \beta_n)\|P_C T z_n - P_C T p\|^2 \\
\leq \beta_n \|x_0 - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\
\leq \beta_n \|x_0 - p\|^2 + (1 - \beta_n)\|x_n - p\|^2 \\
= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2) \\
= \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2(x_n - x_0, p)).
\]

Therefore, \( p \in H_n \) for all \( n \geq 0 \). It means that \( F(T) \subset H_n \) for all \( n \geq 0 \).

Next, we show by mathematical induction that \( F(T) \subset H_n \cap W_n \) for each \( n \geq 0 \). For \( n = 0 \), we have \( W_0 = H \), and hence \( F(T) \subset H_0 \cap W_0 \). Suppose that \( x_i \) is given and \( F(T) \subset H_i \cap W_i \) for some \( i > 0 \). There exists a unique element \( x_{i+1} \in H_i \cap W_i \) such that \( x_{i+1} = P_{H_i \cap W_i}(x_0) \). Therefore, by Lemma 2.2,

\[
\langle x_{i+1} - x_0, p - x_{i+1} \rangle \geq 0
\]

for each \( p \in H_i \cap W_i \). Since \( F(T) \subset H_i \cap W_i \), we get \( F(T) \subset W_{i+1} \). So, we have \( F(T) \subset H_{i+1} \cap W_{i+1} \).

Further, since \( F(T) \) is a nonempty closed convex subset of \( H \), there exists a unique element \( u_0 \in F(T) \) such that \( u_0 = P_{F(T)}(x_0) \). From \( x_{n+1} = P_{H_n \cap W_n}(x_0) \), we obtain

\[
\|x_{n+1} - x_0\| \leq \|z - x_0\|
\]

for every \( z \in H_n \cap W_n \). As \( u_0 \in F(T) \subset W_n \), we get

\[
\|x_{n+1} - x_0\| \leq \|u_0 - x_0\| \quad n \geq 0. \tag{2.1}
\]

This implies that \( \{x_n\} \) is bounded. So, \( \{P_C T P_C(x_n)\}, \{z_n\} \) and \( \{T z_n\} \) are also bounded.

Now, we show that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{2.2}
\]

From the definition of \( W_n \) and Lemma 2.2, it follows that \( x_n = P_{W_n}(x_0) \). As \( x_{n+1} \in H_n \cap W_n \), we have

\[
\|x_{n+1} - x_0\| \geq \|x_n - x_0\| \quad n \geq 0.
\]

Therefore, \( \{\|x_n - x_0\|\} \) is a nondecreasing and bounded sequence. So, there exists \( \lim_{n \to \infty} \|x_n - x_0\| = c \). On the other hand, from \( x_{n+1} \in W_n \), we have
Thus, (2.2) is followed from the last inequality and \( \lim_{n \to \infty} \|x_n - x_0\| = c \).

Since \( \alpha_n \to 1 \) and \( \{x_n\} \) are bounded, we have from (1.7) that

\[
\lim_{n \to \infty} \|z_n - P_C(x_n)\| = \lim_{n \to \infty} (1 - \alpha_n)\|P_C(x_n) - P_CTP_C(x_n)\| = 0. \tag{2.3}
\]

On the other hand, since \( x_{n+1} \in H_n \) we have that

\[
\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \beta_n(\|x_0\| + 2\langle x_n - x_0, x_{n+1} \rangle).
\]

Therefore, from (2.2), the boundedness of \( \{x_n\} \), \( \beta_n \to 0 \) and the last inequality, it follows that

\[
\lim_{n \to \infty} \|y_n - x_{n+1}\| = 0. \tag{2.4}
\]

This together with (2.2) implies that

\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{2.5}
\]

Noticing that \( P_C T z_n = y_n - \beta_n(x_n - P_C T z_n) + \beta_n(x_n - x_0) \), we have

\[
\|x_n - P_C T z_n\| \leq \|x_n - y_n\| + \beta_n\|x_n - P_C T z_n\| + \beta_n\|x_n - x_0\|.
\]

From (2.1) and the last inequality, it follows that

\[
\|x_n - P_C T z_n\| \leq \frac{1}{1 - \beta_n}\left(\|x_n - y_n\| + \beta_n\|u_0 - x_0\|\right).
\]

By \( \beta_n \to 0 \) (\( \beta_n \leq 1 - \beta \) for some \( \beta \in (0, 1) \)), (2.5) and the last inequality, we obtain

\[
\lim_{n \to \infty} \|x_n - P_C T z_n\| = 0. \tag{2.6}
\]

Further, we have that \( P_C T z_n = P_C P_C T z_n \), and hence

\[
\|z_n - P_C T z_n\| \leq \|z_n - P_C(x_n)\| + \|P_C(x_n) - P_C P_C(T z_n)\|
\]

\[
\leq \|z_n - P_C(x_n)\| + \|x_n - P_C T z_n\|.
\]

So, from (2.3), (2.6) and the last inequality, it follows that

\[
\lim_{n \to \infty} \|z_n - P_C T z_n\| = 0. \tag{2.7}
\]
Since \(\{x_n\}\) is bounded, there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) that converges weakly to some element \(p \in H\) as \(j \to \infty\). From (2.6) and (2.7), we also have that \(\{z_{n_j}\}\) converges weakly to \(p\). Since \(\{z_n\} \subset C\), we obtain that \(p \in C\).

By Lemmas 2.3 and (2.7), \(p \in F(T)\) by Lemma 2.6 in [15] with \(S\) replaced by \(T\).

Now, from (2.1) and the weakly lower semicontinuity of the norm it implies that

\[
\|x_0 - u_0\| \leq \liminf_{j \to \infty} \|x_0 - x_{n_j}\| \leq \limsup_{j \to \infty} \|x_0 - x_{n_j}\| \leq \|x_0 - u_0\|.
\]

Thus, we obtain \(\lim_{j \to \infty} \|x_0 - x_{n_j}\| = \|x_0 - u_0\| = \|x_0 - p\|\). This implies \(x_{k_j} \to p = u_0\) by Lemma 2.4. By the uniqueness of the projection \(u_0 = T(x_0)\), we have that \(x_n \to u_0\). From (2.5) and (2.6)-(2.7), we also get \(y_n \to u_0\) and \(z_n \to u_0\), respectively. This completes the proof.

**Corollary 2.6.** Let \(C\) be a nonempty closed convex subset in a real Hilbert space \(H\) and let \(T : C \to H\) be a nonexpansive mapping such that \(F(T) \neq \emptyset\). Assume that \(\{\beta_n\}\) is a sequence in \([0,1]\) such that such that \(\beta_n \to 0\). Then, the sequences \(\{x_n\}\) and \(\{y_n\}\), defined by

\[
x_0 \in H \text{ any element},
\]

\[
y_n = \beta_n x_0 + (1 - \beta_n) PC TP_C(x_n),
\]

\[
H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n(\|x_0\| + 2\langle x_n - x_0, z \rangle)\},
\]

\[
W_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\},
\]

\[
x_{n+1} = P_{H_n \cap W_n}(x_0), n \geq 0,
\]

converge strongly to the same point \(u_0 = T(x_0)\), as \(n \to \infty\).

*Proof.* By putting \(\alpha_n \equiv 1\) in Theorem 2.5, we obtain the conclusion.

**Corollary 2.7.** Let \(C\) be a nonempty closed convex subset in a real Hilbert space \(H\) and let \(T : C \to H\) be a nonexpansive mapping such that \(F(T) \neq \emptyset\). Assume that \(\{\alpha_n\}\) is a sequence in \([0,1]\) such that \(\alpha_n \to 1\). Then, the sequences \(\{x_n\}\) and \(\{y_n\}\), defined by

\[
x_0 \in H \text{ any element},
\]

\[
y_n = PC T(\alpha_n PC(x_n) + (1 - \alpha_n) PC TP_C(x_n)),
\]

\[
H_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\},
\]

\[
W_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\},
\]

\[
x_{n+1} = P_{H_n \cap W_n}(x_0), n \geq 0,
\]

converge strongly to the same point \(u_0 = T(x_0)\), as \(n \to \infty\).
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converge strongly to the same point \( u_0 = P_{F(T)}(x_0) \), as \( n \to \infty \).

**Proof.** By putting \( \beta_n \equiv 0 \) in Theorem 2.5, we obtain the conclusion.

3. Strong convergence to a common fixed point of nonexpansive semigroups

We need the following Lemma in the proof of our result.

**Lemma 3.1** [18]. Let \( C \) be a nonempty bounded closed convex subset in a real Hilbert space \( H \) and let \( \{ T(t) : t > 0 \} \) be a nonexpansive semigroup on \( C \). Then, for any \( h > 0 \)

\[
\limsup_{t \to \infty} \sup_{y \in C} \left\| T(h) \left( \frac{1}{t} \int_0^t T(s)y\,ds \right) - \frac{1}{t} \int_0^t T(s)y\,ds \right\| = 0.
\]

Now, we prove the following result.

**Theorem 3.2.** Let \( C \) be a nonempty closed convex subset in a real Hilbert space \( H \) and let \( \{ T(t) : t > 0 \} \) be a nonexpansive semigroup on \( C \) such that \( \mathcal{F} = \cap_{t > 0} F(T(t)) \neq \emptyset \). Assume that \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences in \([0,1]\) such that \( \alpha_n \to 1 \) and \( \beta_n \to 0 \), and \( \{ t_n \} \) is a positive real divergent sequence. Then, the sequences \( \{ x_n \}, \{ z_n \} \) and \( \{ y_n \} \), defined by (1.8), converge strongly to the same point \( u_0 = P_{F(x_0)} \), as \( n \to \infty \).

**Proof.** For each \( p \in \mathcal{F} \), we have

\[
p = P_C(p) = T(s)P_C(p)
\]

for each \( s > 0 \) and hence from (1.8) and the convexity of \( \| \cdot \|^2 \) we obtain that

\[
\| z_n - p \|^2 = \left\| \alpha_n(P_C(x_n) - p) + (1 - \alpha_n) \left( \frac{1}{t_n} \int_0^{t_n} T(s)P_C(x_n)\,ds - p \right) \right\|^2
\]

\[
= \left\| \alpha_n(P_C(x_n) - P_C(p)) + (1 - \alpha_n) \left( \frac{1}{t_n} \int_0^{t_n} [T(s)P_C(x_n) - T(s)P_C(p)]\,ds \right) \right\|^2
\]

\[
\leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \left( \frac{1}{t_n} \int_0^{t_n} \| T(s)P_C(x_n) - T(s)P_C(p) \| \,ds \right)^2
\]

\[
\leq \alpha_n \| x_n - p \|^2 + (1 - \alpha_n) \| P_C(x_n) - P_C(p) \|^2
\]

\[
\leq \| x_n - p \|^2.
\]
By the similar argument, we also obtain

\[
\|y_n - p\|^2 = \|\beta_n(x_0 - p) + (1 - \beta_n)\left(\frac{1}{t_n} \int_0^{t_n} T(s)z_n ds - p\right)\|^2 \\
\leq \beta_n\|x_0 - p\|^2 + (1 - \beta_n)\left(\frac{1}{t_n} \int_0^{t_n} [T(s)z_n - T(s)p] ds\right)^2 \\
\leq \beta_n\|x_0 - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\
= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2) \\
\leq \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, p \rangle).
\]

Therefore, \( p \in H_n \) for \( n \geq 0 \). It means that \( F \subset H_n \) for \( n \geq 0 \). As in the proof of Theorem 2.5, we can obtain the following properties:

(i) \( F \subset H_n \cap W_n \),

\[
\|x_{n+1} - x_0\| \leq \|u_0 - x_0\|, \quad u_0 = P_F(x_0)
\]

for \( n \geq 0 \). This implies that \( \{x_n\} \) is bounded. So,

\[
\left\{ \frac{1}{t_n} \int_0^{t_n} T(s)P_C(x_n) ds \right\}, \{z_n\} \quad \text{and} \quad \left\{ \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds \right\}
\]

are also bounded.

(ii)

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.2}
\]

\[
\lim_{n \to \infty} \|z_n - P_C(x_n)\| = 0. \tag{3.3}
\]

\[
\lim_{n \to \infty} \|y_n - x_{k+1}\| = 0. \tag{3.4}
\]

\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.5}
\]

\[
\lim_{n \to \infty} \left\| x_n - \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds \right\| = \lim_{n \to \infty} \left\| z_n - \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds \right\| = 0. \tag{3.6}
\]

Since \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) that converges weakly to some element \( p \in H \) as \( j \to \infty \). So, by (3.6), the subsequence \( \{z_{n_j}\} \) also converges weakly to \( p \) and hence \( p \in C \).
On the other hand, for each $h > 0$, we have that
\[
\|T(h)z_n - z_n\| \leq \left\| T(h)z_n - T(h)\left(\frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right)\right\| \\
+ \left\| T(h)\left(\frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right) - \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right\| \\
+ \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds - z_n \right\| \\
\leq 2 \left\| \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds - z_n \right\| \\
+ \left\| T(h)\left(\frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right) - \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right\|.
\]
(3.7)

Let $C_0 = \{ z \in C : \|z - u_0\| \leq 2\|x_0 - u_0\| \}$. Since $u_0 = P_F(x_0) \in C$, we have from (1.8) and (3.1) that
\[
\|z_n - u_0\| = \left\| \alpha_n(P_C(x_n) - u_0) + (1 - \alpha_n)\left[\frac{1}{t_n} \int_0^{t_n} T(s)P_C(x_n) ds - u_0\right]\right\| \\
= \left\| \alpha_n[P_C(x_n) - P_C(u_0)] \right\| \\
+ (1 - \alpha_n)\left[\frac{1}{t_n} \int_0^{t_n} T(s)P_C(x_n) ds - \frac{1}{t_n} \int_0^{t_n} T(s)P_C(u_0) ds \right] \\
\leq \alpha_n\|x_n - u_0\| + (1 - \alpha_n)\left\| \frac{1}{t_n} \int_0^{t_n} [T(s)P_C(x_n) - T(s)P_C(u_0)] ds \right\| \\
\leq \|x_n - x_0\| + \|x_0 - u_0\| \\
\leq 2\|x_0 - u_0\|.
\]

So, $C_0$ is a nonempty bounded closed convex subset. It is easy to verify that and $\{T(t) : t > 0\}$ is a nonexpansive semigroup on $C_0$. By Lemma 3.1, we get
\[
\lim_{n \to \infty} \left\| T(h)\left(\frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right) - \frac{1}{t_n} \int_0^{t_n} T(s)z_n ds\right\| = 0
\]
for every fixed $h > 0$ and hence by (3.6)-(3.7) we obtain that
\[
\lim_{n \to \infty} \|T(h)z_n - z_n\| = 0
\]
for each $h > 0$. By Lemma 2.3, $p \in F(T(h))$ for all $h > 0$. It means that $p \in F$. As in the proof of Theorem 2.5, by using (3.1)-(3.6), we also obtain that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$, defined by (1.8), converge strongly to $u_0$ as $n \to \infty$. This completes the proof.
Corollary 3.3. Let $C$ be a nonempty closed convex subset in a real Hilbert space $H$ and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on $C$ such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in $[0,1]$ such that $\beta_n \to 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by

$$
x_0 \in H \quad \text{any element,}
$$
$$
y_n = \beta_n x_0 + (1 - \beta_n) \frac{1}{t_n} \int_{0}^{t_n} T(s) P_C(x_n) ds,
$$
$$
H_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\
+ \beta_n (\|x_0\| + 2 \langle x_n - x_0, z \rangle)\},
$$
$$
W_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\},
$$
$$
x_{n+1} = P_{H_n \cap W_n}(x_0), n \geq 0,
$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, as $n \to \infty$.

Proof. By putting $\alpha_n \equiv 1$ in Theorem 3.2, we obtain the conclusion.

Corollary 3.4. Let $C$ be a nonempty closed convex subset in a real Hilbert space $H$ and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on $C$ such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n \to 1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by

$$
x_0 \in H \quad \text{any element,}
$$
$$
y_n = \frac{1}{t_n} \int_{0}^{t_n} T(s) \left[ \alpha_n P_C(x_n) + (1 - \alpha_n) \frac{1}{t_n} \int_{0}^{t_n} T(s) P_C(x_n) ds \right] ds,
$$
$$
H_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\},
$$
$$
W_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\},
$$
$$
x_{n+1} = P_{H_n \cap W_n}(x_0), n \geq 0,
$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, as $n \to \infty$.

Proof. By putting $\beta_n \equiv 0$ in Theorem 3.2, we obtain the conclusion.

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References


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