Some Properties of Operator Classes

\((M,k), (M,k)^*, (A,k)\) and \((A,k)^*\)

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Abstract

In this paper we will define a new classes of operators, which we denote by \((A,k)\) and \((A,k)^*\). We give some relations between classes \((M,k), (M,k)^*, (A,k)\) and \((A,k)^*\). Also, some spectral characterizations of \((A,k), (A,k)^*\) classes of operators are given.

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1. Introduction

Let us denote by \(H\) the complex Hilbert space and \(B(H)\) the space of all
bounded linear operators defined in Hilbert space \( H \). In the following we will mention some known classes of operators defined in Hilbert space \( H \). Let \( T \) be an operator in \( B(H) \). The operator \( T \) is called normal if it satisfies the following condition \( T^*T = TT^* \). The operator \( T \) is called quasi-normal if \( T(T^*T) = (T^*T)T \), it is hyponormal if \( T^*T \geq TT^* \), which is equivalent to the condition \( \|Tx\| \geq \|T^*x\| \), for all \( x \) in \( H \). We say that an operator \( T \) is quasi-hyponormal if the following condition \( T^{*2}T^2 \geq (T^*T)^2 \) holds and the last one is equivalent with \( \|T^2x\| \geq \|T^*Tx\| \), for all \( x \) in \( H \). We say that an operator \( T \) is of \((M,k)\) class if \( TT^* \geq (T^*T)^k \), for \( k \geq 2 \), which is equivalent to the condition \( \|Tx\| \geq \|T^*x\|^k \), for all \( x \) in \( H \) and \( k \geq 2 \). It is known that the \((M,2)\) class coincides with the class of quasi-hyponormal operators. But, the class of hyponormal operators does not coincide with \((M,k)\), for any \( k \) (see [3]). We say that an operator \( T \) is of \((M,k)^*\) class if \( T^{*k}T^k \geq (T^*T)^k \), for \( k \geq 1 \), which is equivalent to the condition \( \|T^kx\| \geq \|T^{*k}x\|^k \), for all \( x \) in \( H \) and \( k \geq 1 \) (see [4]). It is known that the \((M,1)^*\) class coincides with the class of hyponormal operators. We will define a new classes of operators which we denote by \((A,k)\) respectively \((A,k)^*\). The operator \( T \) is of \((A,k)\) class if \( \|T^kx\|^2 \geq \|T^*Tx\|^k \), for all unit vectors \( x \) in \( H \) and \( k \geq 2 \). We observe that the \((A,2)\) class coincides with the class of quasi-hyponormal operators. The operator \( T \) is of \((A,k)^*\) class if \( \|T^kx\|^2 \geq \|TT^*x\|^k \), for all unit vectors \( x \) in \( H \) and \( k \geq 1 \). We see that the \((A,2)^*\) class coincides with the \((M,2)^*\) class. The spectrum, the point spectrum, the approximate point spectrum, the joint point spectrum and joint approximate point spectrum of an operator \( T \) are denoted by \( \sigma(T) \), \( \sigma_p(T) \), \( \sigma_{ap}(T) \), \( \sigma_{jp}(T) \), \( \sigma_{ja}(T) \), respectively. We give some relations between classes \((M,k),(M,k)^*,(A,k)\) and \((A,k)^*\). For example, we show that every operator \( T \) from the \((M,k),(k \geq 3)\) class, belongs to the \((A,k)\) class and every operator \( T \) from the \((M,k)^*\) class belongs to the \((A,k)^*\) class. Also, some spectral characterizations of \((A,k)\) \(((A,k)^*)\) class of operators are
Properties of operator classes

Theorem A. (Hölder-McCarthy inequality [2]). Let $A$ be a positive operator. Then the following inequalities hold for all $x$ in $H$

\[\langle A'x, x \rangle \leq \langle Ax, x \rangle^{2(1-r)}, \text{ for } 0 < r \leq 1\]
\[\langle A'x, x \rangle \geq \langle Ax, x \rangle^{2(1-r)}, \text{ for } r \geq 1.\]

Theorem B. (11). Let $\lambda \neq 0$, and $\{x_n\}$ be a sequence of vectors. Then the following assertions are equivalent

i) $(T - \lambda)x_n \to 0$ and $(T^* - \bar{\lambda})x_n \to 0.$

ii) $(|T| - |\lambda|)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0.$

iii) $(|T^*| - |\lambda|)x_n \to 0$ and $(U - e^{-i\theta})x_n \to 0.$

2. Operator classes $(M,k)$ and $(A,k)$ in Hilbert space

In this section we will show some properties of $(M,k)$ and $(A,k)$ classes.

Proposition 2.1. For each positive integer $k \geq 2$ an operator $T$ belongs to the $(M,k)$ class if and only if

\[T^{*k}T^k + 2\lambda(T^*T)^k + \lambda^2 T^{*k}T^k \geq 0,\]

holds for all $\lambda \in R$.

Proof. Let $\lambda \in R$ and $x \in H$ be given. Then $T \in (M,k)$, if and only if

\[\left\| (T^*T)^{\frac{k}{2}} x \right\| \leq \left\| T^k x \right\| \iff 4 \left\| (T^*T)^{\frac{k}{2}} x \right\|^2 - 4 \left\| T^k x \right\|^2 \leq 0\]

\[\iff \left\| T^k x \right\|^2 + 2\lambda \left\| (T^*T)^{\frac{k}{2}} x \right\|^2 + \lambda^2 \left\| T^k x \right\|^2 \geq 0\]

\[\iff \langle (T^k x, T^k x) + 2\lambda \langle (T^*T)^{\frac{k}{2}} x, (T^*T)^{\frac{k}{2}} x \rangle + \lambda^2 \langle T^k x, T^k x \rangle \geq 0\]

\[\iff \langle (T^{*k}T^k + 2\lambda(T^*T)^k + \lambda^2 T^{*k}T^k) x, x \rangle \geq 0\]

\[\iff T^{*k}T^k + 2\lambda(T^*T)^k + \lambda^2 T^{*k}T^k \geq 0.\]

The proof is completed.\[\blacksquare\]
Corollary 2.1. If \( k = 2 \), we get the following relation \( T^2T^2 \geq (T^*T)^2 \) if and only if \( T^2T^2 + 2\lambda(T^*T)^2 + \lambda^2T^*T^2 \geq 0 \), for all \( \lambda \in R \), which is the definition of the quasi-hyponormal operator.

Proposition 2.2. If the operator \( T \) belongs to the \((M, k)\) class, where \( k \geq 2 \), then it belongs to the \((A, k)\) class.

Proof. Let \( T \in (M, k) \), then for every unit vectors \( x \in H \) we have
\[
\|T^k x\|^4 = \langle T^k x, T^k x \rangle^2 = \langle T^{2k} T^k x, x \rangle^2 \\
\geq \langle (T^*T)^k x, x \rangle^2, \text{ because } T \in (M, k) \\
\geq \langle (T^*T)^2 x, x \rangle^k \text{ (Hölder-McCarthy inequality)} \\
= \langle T^*T x, T^*T x \rangle^k = \|T^*T x\|^{2k}.
\]
Thus \( \|T^k x\|^4 \geq \|T^*T x\|^{2k} \), respectively \( T \in (A, k) \). Therefore the proof is completed.

Corollary 2.2. If the operator \( T \) belongs to the \((M, k)\)' class, where \( k \geq 1 \), then \( T \) belongs to the \((A, k+1)\) class.

Proof. This proof follows from theorem 3.8 in [3] and proposition 2.2.

Proposition 2.3. Let \( T \) be an operator in \((A, k)\). Then the following relation
\[
0 \neq \lambda \in \sigma_p(T) \Rightarrow \tilde{\lambda} \in \sigma_p(T^*),
\]
holds.

Proof. Let \( T \in (A, k) \), \( \lambda \in \sigma_p(T), \lambda \neq 0 \) and let \( x \in H \) a unit vector such that \( Tx = \lambda x \). Then we have
\[
\|T^* T x\| \leq \|T^k x\|^2 \\
\|T^* T x\| = |\lambda|^k \|T^* x\|^2 \\
\|T^k x\|^2 = |\lambda|^{2k}.
\]
Now from relation (1), (2) and (3) it follows that
\[
\|T^* x\| \leq |\lambda|
\]
Hence
\[
\|T^* x - \tilde{\lambda} x\|^2 = \langle T^* x - \tilde{\lambda} x, T^* x - \tilde{\lambda} x \rangle = \langle T^* x, T^* x \rangle - \langle T^* x, \tilde{\lambda} x \rangle - \langle \tilde{\lambda} x, T^* x \rangle + \langle \tilde{\lambda} x, \tilde{\lambda} x \rangle \\
= \|T^* x\|^2 - \tilde{\lambda} \langle x, Tx \rangle - \tilde{\lambda} \langle Tx, x \rangle + |\lambda|^2 \langle x, x \rangle \\
\leq |\lambda|^2 - |\lambda|^2 \|x\|^2 - |\lambda|^2 \|x\|^2 + |\lambda|^2 \|x\|^2 = |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0.
\]
Therefore we have \( T^* x = \tilde{\lambda} x \), consequently \( \tilde{\lambda} \in \sigma_p(T^*) \).
Lemma 2.1. If $T$ is a bilateral weighted shift operator, with weighted sequence $\omega_n$, $(Te_n = \omega_n e_{n+1})$, then $T$ is in $(A,k)$ class if and only if

$$|\omega_n| \cdot |\omega_{n+1}| \cdots |\omega_{n+k-1}| \geq |\omega_n|^k, \quad n \in \mathbb{Z} \text{ and } k \geq 2.$$  

Proof. The proof follows immediately from the definition of $(A,k)$ class. $\blacksquare$

Lemma 2.2. If $T$ is a regular bilateral weighted shift operator, with weighted sequence $\omega_n \neq 0$, $(Te_n = \omega_n e_{n+1})$, then $T^{-1} \in (A,k)$ if and only if

$$|\omega_{n-1}| \cdot |\omega_{n-2}| \cdots |\omega_{n-k}|, \quad n \in \mathbb{Z}, |\omega_n| \neq 0 \text{ and } k \geq 2.$$  

Proof. The proof follows immediately from the definition of $(A,k)$ class. $\blacksquare$

By the following example we show that there exists an operator $T$ which belongs to the $(A,k)$ class but its inverse $T^{-1}$ is not element of $(A,k)$.

Example 2.1. Let $T$ is a regular bilateral weighted shift operator, with weighted sequence $(\omega_n)$ given as follows

$$\omega_n = \begin{cases} 
\frac{1}{2}, & \text{for } n \leq -1 \\
1, & \text{for } n = 0 \\
\frac{1}{2}, & \text{for } n = 1 \\
2, & \text{for } n = 2 \\
\frac{1}{2}, & \text{for } n = 3 \\
16, & \text{for } n \geq 4 
\end{cases}.$$  

From lemma 2.1 and lemma 2.2, after some calculations it follows that $T \in (A,3)$, but $T^{-1} \notin (A,3)$.

Theorem 2.1. Let $T \in (A,k)$ and let $T^n, (n > k)$ be a compact operator, then it follows that $T$ is compact too.

Proof. Let $T \in (A,k)$, $k \geq 2$ and let $\frac{T^{n+k}x}{\|T^{n+k}x\|}$ be a unit vector in Hilbert space $H$. Then we have

$$\left\|T^*T \left(\frac{T^{n+k}x}{\|T^{n+k}x\|}\right)\right\|^k \leq \|T^nx\|^2 \cdot \left\|T^{n-k}x\right\|^{-2}.$$  

(5)
Let \((x_n) \in H\) be weakly convergent sequence with limit 0 in \(H\). From compactness of \(T^n\) and inequality (5) we get the following relation
\[
\left\| T^{n-k} x_m \right\| \leq \left\| T^k x_m \right\|^2 \to 0, n \to \infty.
\]
So, the operator \(T^{n-k}\) is a compact operator from which follows that \(T^{n-1}\) is also compact, respectively \(T^n\) is a compact operator. Now, if we repeat this procedure, we conclude the \(T\) is a compact operator.

**Proposition 2.4.** Let \(T \in (A,k)\) be a regular operator, for \(k \geq 2\). Then the approximate point spectrum lies in the disc
\[
\sigma_{ap} \subseteq \{ \lambda \in C : \frac{1}{\|T\|^{k-1} \sqrt{(T^T)^{-1} T}} \leq |\lambda| \leq \|T\| \}.
\]

**Proof.** Let \(T \in (A,k)\) be a regular operator, for \(k \geq 2\). Then for every unit vector \(x\) in \(H\), we have
\[
\|x\|^k = \left\| (T^T)^{-1} (T^T)x \right\|^k \leq \left\| (T^T)^{-1} \right\|^k \cdot \left\| T^T x \right\|^k \leq \left\| (T^T)^{-1} \right\|^k \cdot \left\| T^k x \right\|^2 \leq \left\| (T^T)^{-1} \right\|^k \cdot \left\| T^k \right\|^2.
\]

\[
\|Tx\| \geq \frac{1}{\|T^k\| \sqrt{\|T^T\|^{-1}}} \geq \frac{1}{\|T\|^{k-1} \sqrt{\|T^T\|}}.
\]

(6)

Now, assume that \(\lambda \in \sigma_{ap}\). Then there exists a sequence \((x_n)\), \(\|x_n\| = 1\), such that \(\| (T - \lambda)x_n \| \to 0\), when \(n \to \infty\). Therefore by (6) we have
\[
\|Tx_n - \lambda x_n\| \geq \|Tx_n\| - |\lambda| \|x_n\| \geq \frac{1}{\|T\|^{k-1} \sqrt{(T^T)^{-1}}} - |\lambda|.
\]

(7)

Now, when \(n \to \infty\), from relation (7) we have
\[
|\lambda| \geq \frac{1}{\|T\|^{k-1} \sqrt{(T^T)^{-1}}}.
\]

respectively \(\sigma_{ap} \subseteq \{ \lambda \in C : \frac{1}{\|T\|^{k-1} \sqrt{(T^T)^{-1}}} \leq |\lambda| \leq \|T\| \} .
\]

**Corollary 2.3.** Let \(T\) be a regular quasi-hyponormal operator. Then the following relation
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\[ \sigma_{ap} \subseteq \left\{ \lambda \in \mathbb{C} : \frac{1}{\|T\| \cdot \|T^*T\|^{-1}} \leq |\lambda| \leq \|T\| \right\} \]

holds.

**Proposition 2.5.** If the operator \( T \in B(H) \) satisfies condition \( |T^k|^\frac{4}{k} \geq |T|^4 \), for \( k \geq 2 \), then operator \( T \) belongs to the \((A,k)\) class.

**Proof.** Assume that the operator \( T \) satisfies the condition

\[ |T^k|^\frac{4}{k} \geq |T|^4 , \text{ for } k \geq 2 . \]

Then, for every unit vector \( x \in H \), we have

\[ \left\| T^k x \right\|^4 = \langle T^k x, T^k x \rangle^2 = \langle T^{2k} x, x \rangle^2 \]

\[ \geq \langle T^k x, x \rangle^k \quad (\text{Hölder-McCarthy inequality}) \]

\[ \geq \langle |T|^4 x, x \rangle^k = \langle (T^*T)^2 x, x \rangle^k = \langle T^*T x, T^*T x \rangle^k = \|T^*T\|^2 \cdot \|T\|^4 . \]

Thus \( \left\| T^k x \right\|^4 \geq \|T^*T\|^2 \), for \( k \geq 2 \), respectively \( T \in (A,k) \). Therefore the proof is completed. ■

**Corollary 2.4.** If the operator \( T \in B(H) \) satisfies condition \( |T^2|^2 \geq |T|^4 \), then it is quasi-hyponormal.

**Theorema 2.2.** If \( T \in B(H) \) belongs to the class \((A,k), k \geq 2 \) and \( \lambda \neq 0 \), then for a sequence of unit vectors \( \{ x_n \} \), \( (T - \lambda)x_n \to 0 \) implies

\[ (T^* - \bar{\lambda})x_n \to 0, \quad \sigma_p(T) - \{0\} = \sigma_{sp}(T) - \{0\} \quad \text{and} \quad \sigma_{ap}(T) - \{0\} = \sigma_{sp}(T) - \{0\} . \]

**Proof.** We only need to prove that \( (T^* - \bar{\lambda})x_n \to 0 \) by Theorem B. By the assumption \( (T - \lambda)x_n \to 0 \) and \( (T^k - \lambda^k)x_n \to 0 \), we have

\[ \|Tx_n\| \to |\lambda| \quad \text{and} \quad \|T^k x_n\| \to |\lambda|^k . \]
Since $T \in B(H)$ belongs to the $(A,k)$ class, then we have

$$\|T^2 x_n\| = \|T^* T x_n\| \leq \|T^4 x_n\|^4 \Rightarrow (\lambda^4)^{\frac{4}{k}} = |\lambda|^4.$$ 

Thus

$$\|T^4 x_n\| \Rightarrow |\alpha| \leq |\lambda|^4.$$ 

Relation (9)

Now, from relation (9) we have:

$$\|\left(T^2 - |\lambda|^2\right)x_n\|^2 = \langle \left(T^2 - |\lambda|^2\right)x_n, \left(T^2 - |\lambda|^2\right)x_n \rangle$$

$$= \langle T^2 x_n, T^2 x_n \rangle - \langle T^2 x_n, |\lambda|^2 x_n \rangle - \langle |\lambda|^2 x_n, T^2 x_n \rangle + \langle |\lambda|^2 x_n, |\lambda|^2 x_n \rangle$$

$$= \|T^2 x_n\|^2 - |\lambda|^4 \langle T x_n, T x_n \rangle - |\lambda|^4 \langle T x_n, T x_n \rangle + |\lambda|^4 \langle x_n, x_n \rangle,$$

$$= \|T^2 x_n\|^2 - 2|\lambda|^4 \|T x_n\|^2 + |\lambda|^4$$

$$\Rightarrow |\alpha| - 2|\lambda|^2 \|T x_n\|^2 + |\lambda|^4 \leq |\lambda|^4 - 2|\lambda|^4 + |\lambda|^4 = 0.$$ 

respectively

$$\|\left(T^2 - |\lambda|^2\right)x_n\|^2 \Rightarrow 0.$$ 

Relation (10)

From the relation (10) we have

$$\left(\left|T\right| - |\lambda|\right)x_n = \left(\left|T\right| + |\lambda|\right)^{-1} \cdot \left(\left|T\right|^2 - |\lambda|^2\right)x_n \Rightarrow 0.$$ 

Thus

$$\left(U - e^{i\theta}\right)\lambda x_n = U\left(\left|\lambda\right| - |T|\right)x_n + \left(U\left|T\right| - e^{i\theta}|\lambda|\right)x_n \Rightarrow 0.$$ 

So, that $\left(T^*-\lambda\right)x_n \Rightarrow 0$, by Theorem B. Therefore the proof is completed. 

3. Operator classes $(M,k)^*$ and $(A,k)^*$ in Hilbert space

In this section we will show some properties of $(M,k)^*$ and $(A,k)^*$ classes.

**Proposition 3.1.** If the operator $T$ belongs to the class $(A,1)^*$, then it is hyponormal operator.

**Proof.** Let $T \in (A,1)^*$, then for every unit vectors $x \in H$ we have
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\[ \|T^k x\|^2 \geq \|TT^* x\| \]

\[ \langle Tx, Tx \rangle \geq \langle TT^* x, TT^* x \rangle \frac{1}{2} \]

\[ \langle Tx, Tx \rangle - \langle TT^* x, TT^* x \rangle \frac{1}{2} \geq 0 \]

\[ \langle T^k x, x \rangle - \langle (TT^*)^k x, x \rangle \frac{1}{2} \geq 0, \text{(Hölder-McCarthy inequality)} \]

\[ \langle (T^k - TT^*) x, x \rangle \geq 0. \]

Thus \( T^k T \geq TT^* \), respectively \( T \) is hyponormal operator. ■

**Corollary 3.1.** If the operator \( T \) belongs to the \((A,1)^*\) class, then it belongs to the \((M,1)^*\) class.

**Proposition 3.2.** If the operator \( T \) belongs to the \((M,k)^*\) class, for \( k \geq 3 \), then it belongs to the \((A,k)^*\) class.

**Proof.** Let \( T \in (M,k)^* \), then for every unit vectors \( x \in H \) we have :

\[ \|T^k x\|^2 = \langle T^k x, T^k x \rangle = \langle (T^* T)^k x, x \rangle \]

\[ \geq \langle (TT^*)^k x, x \rangle, \text{ because } T \in (M,k)^* \]

\[ \geq \langle (TT^*)^2 x, x \rangle^k \text{ (Hölder-McCarthy inequality)} \]

\[ = \langle TT^* x, TT^* x \rangle^k = \|TT^* x\|^2^k. \]

Thus \( T \in (A,k)^* \). Therefore the proof is completed. ■

**Corollary 3.2.** If the operator \( T \) have dense range in Hilbert space and belongs to the \((M,k+1)^*\) class, for \( k \geq 2 \), then it belongs to the \((A,k)^*\) class.

**Proof.** This proof follows from theorem 3.8 in [3] and proposition 3.2. ■

**Lemma 3.1.** If \( T \) is bilateral weighted shift operator, with weighted sequence \( \omega_n \), \( (Te_n = \omega_n e_{n+1}) \), then it is \((A,k)^*\) class if and only if

\[ |\omega_n| \cdot |\omega_{n+1}| \cdots |\omega_{n+k-1}| \geq |\omega_{n+1}|^k, \quad n \in \mathbb{Z} \text{ and } k \geq 1. \]
Proof. This follows immediately from the definition of \((A,k)^*\) class.

**Example 3.1.** Let \(T\) is a bilateral weighted shift operator, with weighted sequence \((\omega_n)\) given as follows

\[
\omega_n = \begin{cases} 
\frac{1}{5} & \text{per } n \leq -1 \\
\frac{5}{1} & \text{per } n = 0 \\
\frac{1}{5} & \text{per } n = 1 \\
25 & \text{per } n \geq 2 
\end{cases}
\]

From lemma 2.1 and lemma 3.1, after some calculation it follows that \(T \in (A,3)^*\), but \(T \not\in (A,3)\).

**Example 3.2.** Let \(T\) is a bilateral weighted shift operator, with weighted sequence \((\omega_n)\) given as follows

\[
\omega_n = \begin{cases} 
\frac{1}{2} & \text{for } n \leq -1 \\
1 & \text{for } n = 0 \\
\frac{1}{2} & \text{for } n = 1 \\
2 & \text{for } n = 2 \\
\frac{1}{4} & \text{for } n = 3 \\
16 & \text{for } n \geq 4 
\end{cases}
\]

From lemma 2.1 and lemma 3.1, after some calculation it follows that \(T \in (A,3)\), but \(T \not\in (A,3)^*\).

**Lemma 3.2.** If \(T\) is a regular bilateral weighted shift operator, with weighted sequence \(\omega_n \neq 0, (Te_n = \omega_n e_{n+1})\), then \(T^{-1} \in (A,k)^*\) if and only if

\[
|\omega_n| \geq |\omega_{n-1}| \cdot |\omega_{n-2}| \ldots |\omega_{n-k}|, \quad n \in \mathbb{Z}, |\omega_n| \neq 0 \text{ and } k \geq 1.
\]

Proof. This follows immediately from the definition of \((A,k)^*\) class.

**Proposition 3.3.** Let \(T \in (A,k)^*\) be a regular operator, for \(k \geq 1\). Then the approximate point spectrum lies in the disc...
Properties of operator classes

\[ \sigma_{ap} \subseteq \left\{ \lambda \in C : \frac{1}{\|T\|^{k-1} \sqrt{\|(TT^*)^{-1}\|^k}} \leq |\lambda| \leq \|T\| \right\}. \]

**Proof.** The proof of the Proposition is similar to that of Proposition 2.4. ■

**Proposition 3.4.** If the operator \( T \in B(H) \) satisfies condition \( \|T^k\|^4 \geq \|T^*\|^4 \), for \( k \geq 2 \), then operator \( T \) belongs to the \((A,k)^*\) class.

**Proof.** The proof of the Proposition is similar to that of Proposition 2.5. ■

**Theorem 3.1.** If \( T \in (A,k)^* \), then \( T \) is normaloid for every positive integer \( k \geq 1 \).

**Proof.** Suppose \( T \) belongs to the \((A,k)^*\) class, for \( k \geq 1 \). Then we have

\[ \|TT^*x\|^k \leq \|T^k\|^2, \|x\| = 1 \]

\[ \sup_{||x||=1} \|TT^*x\|^k \leq \sup_{||x||=1} \|T^k\|^2 \]

\[ \|TT^*\|^k \leq \|T\|^k. \]

Now, since \( \|T\|^2 = \|T^*\|^2 = \|T^*T\| = \|TT^*\| \), we have \( \|T\|^{2k} \leq \|T^k\|^2 \), respectively \( \|T\|^k \leq \|T^*\| \). Therefore

\[ \|T^k\| = \|T\|^k. \] (11)

Now, we prove the equality (11) for \( k + 1 \). Since \( \frac{TT^*x}{\|TT^*x\|} \) is a unit vector, then we have

\[ \|TT^*T^*x\|^k \leq \|T^kT^*x\|^2 \cdot \|TT^*x\|^{k-2} \Rightarrow \|TT^*\|^2 \leq \|T^k\|^2 \cdot \|T^*\|^2 \Rightarrow \|TT^*\|^{2k} \leq \|T^k\|^{2k} \cdot \|T^*\|^{2k} \Rightarrow \]

\[ \|T\|^{2k+2} \leq \|T^k\|^2 \Rightarrow \|T\|^{k+1} \leq \|T^k\| \leq \|T\|^{k+1}. \]

Therefore
\[ \|T^{k+1}\| = \|T\|^{k+1}. \]  
(12)

Now from the relation (12) it follows that \( \|T^k\| = \|T\|^k \), for every positive integer \( k \geq 1 \). Therefore

\[
r(T) = \lim_{k \to \infty} \left( \|T^K\|^{\frac{1}{k}} \right) = \lim_{k \to \infty} \|T\|^\frac{1}{k} = \|T\|,
\]
and the proof is completed. ■

In the following we denote by \( (QD)(P_n) \) the class of quasi-diagonal operator with respect to the sequence \( (P_n)_{n \in N} \) of orthogonal projections such that \( P_n \to 1 \), strongly (see [7]).

**Theorem 3.2.** If \( T^*T = \lambda I + K \), were \( \lambda \in \mathbb{C} \), \( K \) is a compact operator, and \( T^* \) is a quasi-normal, then \( TT^* = \beta I - K'' \) for some complex number \( \beta \) and some compact operator \( K'' \).

**Proof.** For \( \alpha > \|T^*T\| = \|T\|^2 \) we have

\[
(\alpha I - T^*T)^{-1} = \frac{1}{\alpha} + \frac{1}{\alpha} T^* (\alpha I - TT^*)^{-1} T
\]

\[
(\alpha I - T^*T)^{-1} = \frac{1}{\alpha} + \frac{1}{\alpha} (\alpha I - TT^*)^{-1} T^* T.
\]  
(13)

Since \( T^*T \) is a Fredholm operator and \( \text{ind} T^*T = 0 \), it follows that there exist a compact operator \( K \), such that \( T^*T + K \) is an invertible operator. Now by (13), we have

\[
(\alpha I - T^*T)^{-1} = \frac{1}{\alpha} + \frac{1}{\alpha} (\alpha I - TT^*)^{-1} (T^*T + K) - \frac{1}{\alpha} (\alpha I - TT^*)^{-1} K
\]

were \( K_1 = -\frac{1}{\alpha} (\alpha I - TT^*)^{-1} K \) is a compact operator,

\[
(\alpha I - T^*T)^{-1} = \frac{1}{\alpha} + \frac{1}{\alpha} (\alpha I - TT^*)^{-1} (T^*T + K) + K_1
\]

\[
\frac{1}{\alpha} (\alpha I - TT^*)^{-1} = \left[ (\alpha I - T^*T)^{-1} - K_1 - \frac{I}{\alpha} \right] (T^*T + K)^{-1}
\]

\[
(\alpha I - TT^*)^{-1} = \alpha \left[ (\alpha I - T^*T)^{-1} - K_1 - \frac{I}{\alpha} \right] (T^*T + K)^{-1}.
\]

So, \( (\alpha I - TT^*)^{-1} \in (QD)(P_n) \), for every sequence \( (P_n)_{n \in N} \) of orthogonal
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projections, for which $P_n \to 1$ strongly. Therefore $\alpha I - TT^* = \gamma I + K''$ (see [7]), respectively $TT^* = (\alpha - \gamma)I - K'' = \beta I - K^*$. ■

**Proposition 3.5.** Let $T \in (A,k)^*$ and $T^k$ be a compact operator for some $k \geq 1$, then $T$ is compact too.

**Proof.** From the fact that $T \in (A,k)^*$ for $k \geq 1$, we have

$$\|TT^*x\| \leq \|T^k x\|^2,$$

for every $x \in H, \|x\| = 1$ and $k \geq 1$. (14)

Let $(x_n) \in H$ be weakly convergent sequence with limit 0 in $H$. From compactness of $T^k$ and relation (14) we get the following relation

$$\|TT^*x_n\| \leq \|T^k x_n\|^2 \to 0, n \to \infty$$

From the last relation it follows that $TT^*$ is a compact operator, respectively $T$ is a compact operator. ■

**REFERENCES**


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