Strongly $g^*$-Closed Sets in Topological Spaces

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Abstract

In this paper, the authors introduce and investigate the concept of strongly generalized $g^*$-closed sets (briefly strongly $g^*$-closed set) and investigate the relation between the associated topology.

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1 Introduction

Levine (1960) introduced the notion of generalized closed (briefly g-closed) sets in topological spaces and showed that compactness, countably compactness, para compactness and normality etc are all g-closed hereditary. Andrijevic (1986), Arya and Nour (1990), Bhattacharya and Lahiri (1987), Dontchev (1995, 1996), Ganambal (1997), Levine (1960, 1963), Maki (1993, 1994, 1996), Mashhour et. al. (1982), Njastad (1965), Palaniappan (1993), Velicko (1968) and Veerakumar (2000) introduced and investigated semi-preopen sets, generalized semiopen sets, semi-generalized open sets, generalized semi-preopen sets, $\delta$ -generalized closed sets, $\theta$ -generalized closed sets, pre regular closed sets, generalized open sets, semi open sets, $\alpha$-closed sets, regular generalized closed sets, H-closed sets and $g^*$ -closed sets which are some of the weak and stronger form of open sets and complements of these sets are called the same type g-closed sets respectively.
Veerakumar (2000) introduced and investigated between closed sets and \( g^* \)-closed sets. The aim of this paper is to introduce and study stronger form of generalized \( g^* \)-closed sets in a topological space. Also we investigate topological properties of strongly \( g^* \)-closed sets. Throughout this paper \((X, \tau)\) represent non empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset \( A \) of \((X, \tau)\), \( \text{cl}(A) \) and \( \text{int}(A) \) represent the closure of \( A \) with respect to \( \tau \) and the interior of \( A \) with respect to \( \tau \) respectively.

2 Preliminaries

Before entering into our work, we recall the following definitions which are due to Levine.

Definition 2.1. [13]: A subset \( A \) of a topological space \((X, \tau)\) is called a pre-open set if \( A \subseteq \text{int}(\text{cl}(A)) \) and pre-closed set if \( \text{cl}(\text{int}(A)) \subseteq A \).

Definition 2.2. [8] A subset \( A \) of a topological space \((X, \tau)\) is called a semi-open set if \( A \subseteq \text{cl}(\text{int}(A)) \) and semi closed set if \( \text{int}(\text{cl}(A)) \subseteq A \).

Definition 2.3. [14] A subset \( A \) of a topological space \((X, \tau)\) is called an \( \alpha \)-open set if \( A \subseteq \text{int}(\text{cl}(\text{int}(A))) \) and an \( \alpha \)-closed set if \( \text{cl}(\text{int}(\text{cl}(A))) \subseteq A \).

Definition 2.4. [1] A subset \( A \) of a topological space \((X, \tau)\) is called a semi pre-open set (\( \beta \)-open set) if \( A \subseteq \text{cl}(\text{int}(\text{cl}(A))) \) and semi-preclosed set if \( \text{int}(\text{cl}(\text{int}(A))) \subseteq A \).

Definition 2.5. [16] A subset \( A \) of a topological space \((X, \tau)\) is called a \( \delta \)-closed set if \( A = \text{cl}_\delta(A) \) where \( \text{cl}_\delta(A) = \{x \in X : \text{int}(\text{cl}(U)) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U\} \).

Definition 2.6. [16] A subset \( A \) of a topological space \((X, \tau)\) is called a \( \theta \)-closed set if \( A = \text{cl}_\theta(A) \) where \( \text{cl}_\theta(A) = \{x \in X : \text{cl}(U) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U\} \).

Definition 2.7. [9] A subset \( A \) of a topological space \((X, \tau)\) is called a \( g \)-closed if \( \text{cl}(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

Definition 2.8. [3] A subset \( A \) of a topological space \((X, \tau)\) is called a semi-generalized closed set (briefly \( \text{sg} \)-closed) if \( \text{scl}(A) \subseteq U \), whenever \( A \subseteq U \), \( U \) is semi-open in \((X, \tau)\).

Definition 2.9. [2] A subset \( A \) of a topological space \((X, \tau)\) is called a generalized semi-closed set (briefly \( \text{gs} \)-closed) if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \), \( U \) is open in \((X, \tau)\).
Definition 2.10. [11] A subset $A$ of a topological space $(X, \tau)$ is called a generalized $\alpha$-closed (briefly $g\alpha$-closed) if $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$.

Definition 2.11. [10] A subset $A$ of a topological space $(X, \tau)$ is called an $\alpha$-generalized closed set (briefly $\alpha g$-closed) if $\alphacl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

Definition 2.12. [14] A subset $A$ of a topological space $(X, \tau)$ is called a generalized semi pre-closed set (briefly $gsp$-closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

Definition 2.13. [154] A subset $A$ of a topological space $(X, \tau)$ is called a regular generalized closed set (briefly r-g-closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.

Definition 2.14. [12] A subset $A$ of a topological space $(X, \tau)$ is called a generalized pre closed set (briefly $gp$-closed) if $pcl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

Definition 2.15. [7] A subset $A$ of a topological space $(X, \tau)$ is called a generalized pre regular closed set (briefly gpr-closed) if $pcl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is regular open in $(X, \tau)$.

Definition 2.16. [6] A subset $A$ of a topological space $(X, \tau)$ is called a $\theta-$generalized closed set (briefly $\theta g$-closed) if $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

Definition 2.17. [5] A subset $A$ of a topological space $(X, \tau)$ is called a $\delta$-generalized closed set (briefly $\delta g$ closed) if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

Definition 2.18. [17] A subset $A$ of a topological space $(X, \tau)$ is called a $g^*$-closed set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $(X, \tau)$.

3 Strongly $g^*$-closed sets

In this section we have introduce the concept of strongly $g^*$-closed sets in topological space and we investigate the group of structure of the set of all strongly $g^*$-closed sets.

Definition 3.1. Let $(X, \tau)$ be a topological space and $A$ be its subset, then $A$ is strongly $g^*$-closed set if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open.

Theorem 3.1. Every closed set is strongly $g^*$-closed set.
Proof. The proof is immediate from the definition of closed set.

Example 3.1. The converse of the above theorem need not be true from the following example.
Let \( X = \{ a, b, c \} \) and \( \tau = \{ \emptyset, X, \{ a \}, \{ a, c \} \} \). Let \( A = \{ a, b \} \). \( A \) is a strongly \( g^* \)-closed set but not a closed set of \( (X, \tau) \).

Theorem 3.2. If a subset \( A \) of a topological space \( X \) is \( g^* \)-closed then it is strongly \( g^* \)-closed in \( X \) but not conversely.

Proof. Suppose \( A \) is \( g^* \)-closed in \( X \). Let \( G \) be an open set containing \( A \) in \( X \). Then \( G \) contains \( \text{cl}(A) \). Now \( G \supseteq \text{cl}(A) \supseteq \text{cl}(\text{int}(A)) \). Thus \( A \) is strongly \( g^* \)-closed in \( X \).

Example 3.2. The converse of the above theorem need not be true as seen from the following example.
Let \( X = \{ a, b, c \} \) with topology \( \tau = \{ X, \emptyset, \{ a \}, \{ a, b \} \} \). In this topological space the subset \( \{ b \} \) is strongly \( g^* \)-closed but not \( g^* \)-closed set.

Theorem 3.3. If \( A \) is a subset of a topological space \( X \) is open and strongly \( g^* \)-closed then it is closed.

Proof. Suppose a subset \( A \) of \( X \) is both open and strongly \( g^* \)-closed. Now \( A \supseteq \text{cl}(\text{int}(A)) \supseteq \text{cl}(A) \). Therefore \( A \supseteq \text{cl}(A) \). Since \( \text{cl}(A) \supseteq A \). We have \( A \subseteq \text{cl}(A) \). Thus \( A \) is closed in \( X \).

Corollary 3.1. If \( A \) is both open and strongly \( g^* \)-closed in \( X \) then it is both regular open and regular closed in \( X \).

Proof. As \( A \) is open \( A = \text{int}(A) = \text{int}(\text{cl}(A)) \), since \( A \) is closed. Thus \( A \) is regular open. Again \( A \) is open in \( X \), \( \text{cl}(\text{int}(A)) = \text{cl}(A) \). As \( A \) is closed \( \text{cl}(\text{int}(A)) = A \). Thus \( A \) is regular closed.

Corollary 3.2. If \( A \) is both open and strongly \( g^* \)-closed then it is \( rg \)-closed.

Theorem 3.4. If a subset \( A \) of a topological space \( X \) is both strongly \( g^* \)-closed and semi open then it is \( g^* \)-closed.

Proof. Suppose \( A \) is both strongly \( g^* \)-closed and semi open in \( X \), Let \( G \) be an open set containing \( A \). As \( A \) is strongly \( g^* \)-closed, \( G \supseteq \text{cl}(\text{int}(A)) \). Now \( G \supseteq \text{cl}(A) \). since \( A \) is semi open. Thus \( A \) is \( g^* \)-closed in \( X \).

Corollary 3.3. If a subset \( A \) of a topological space \( X \) is both strongly \( g^* \)-closed and open then it is \( g^* \)-closed set.

Proof. As every open set is semiopen by the above theorem the proof follows.
**Theorem 3.5.** A set $A$ is strongly $g^*$-closed iff $cl(int(A)) - A$ contains no non empty closed set.

**Proof.** **Necessary:** Suppose that $F$ is non empty closed subset of $cl(int(A))$. Now $F \subseteq cl(int(A)) - A$ implies $F \subseteq cl(int(A)) \cap A^c$, since $cl(int(A)) - A = cl(int(A)) \cap A^c$. Thus $F \subseteq cl(int(A))$. Now $F \subseteq A^c$ implies $A \subseteq F^c$. Here $F^c$ is $g$-open and $A$ is strongly $g^*$-closed, we have $cl(int(A)) \subseteq F^c$. Thus $F \subseteq (cl(int(A)))^c$. Hence $F \subseteq (cl(int(A))) \cap (cl(int(A)))^c = \emptyset$. Therefore $F = \emptyset \Rightarrow cl(int(A)) - A$ contains no non empty closed sets.

**Sufficient:** Let $A \subseteq G$, $G$ is $g$-open. Suppose that $cl(int(A))$ is not contained in $G$ then $(cl(int(A)))^c$ is a non empty closed set of $cl(int(A)) - A$ which is a contradiction. Therefore $cl(int(A)) \subseteq G$ and hence $A$ is strongly $g^*$-closed. □

**Corollary 3.4.** A strongly $g^*$-closed set $A$ is regular closed iff $cl(int(A)) - A$ is closed and $cl(int(A)) \subseteq A$.

**Proof.** Assume $A$ that $A$ is regular closed. Since $cl(int(A)) = A$, $cl(int(A)) - A = \emptyset$ is regular closed and hence closed.

Conversely assume that $cl(int(A)) - A$ is closed. By the above theorem $cl(int(A)) - A$ contains no nonempty closed set. Therefore $cl(int(A)) - A = \emptyset$. Thus $A$ is regular closed. □

**Theorem 3.6.** Suppose that $B \subseteq A \subseteq X$, $B$ is strongly $g^*$-closed set relative to $A$ and that both open and strongly $g^*$-closed subset of $X$ then $B$ is strongly $g^*$-closed set relative to $X$.

**Proof.** Let $B \subseteq G$ and $G$ be an open set in $X$. But given that $B \subseteq A \subseteq X$, therefore $B \subseteq A$ and $B \subseteq G$. This implies $B \subseteq A \cap G$. Since $B$ is strongly $g^*$-closed relative to $A$, $cl(int(B)) \subseteq A \cap G$. This implies $A \cap (cl(int(B))) \subseteq G$. Thus $A \cap (cl(int(B))) \cup (cl(int(B)))^c \subseteq G \cup (cl(int(B)))^c$. Since $A$ is strongly $g^*$-closed in $X$, we have $(cl(int(A))) \subseteq G \cup (cl(int(B)))^c$. Also $B \subseteq A \Rightarrow cl(int(B)) \subseteq cl(int(A))$. Thus $cl(int(B)) \subseteq cl(int(A)) \subseteq G \cup (cl(int(B)))^c$. Therefore $B$ is strongly $g^*$-closed set relative to $X$. □

**Corollary 3.5. Corollary 3.14:** Let $A$ be strongly $g^*$-closed and suppose that $F$ is closed then $A \cap F$ is strongly $g^*$-closed set.

**Proof.** To show that $A \cap F$ is strongly $g^*$-closed, we have to show $cl(int(A \cap F)) \subseteq G$ whenever $A \cap F \subseteq G$ and $G$ is $g$-open. $A \cap F$ is closed in $A$ and so strongly $g^*$-closed in $B$. By the above theorem $A \cap F$ is strongly $g^*$-closed in $X$. Since $A \cap F \subseteq A \subseteq X$. □

**Theorem 3.7. Theorem 3.15:** If $A$ is strongly $g^*$-closed and $A \subseteq B \subseteq cl(int(A))$ then $B$ is strongly $g^*$-closed.
Proof. Given that $B \subseteq \text{cl} (\text{int} (A))$ then $\text{cl} (\text{int} (B)) \subseteq \text{cl} (\text{int} (A)), \text{cl} (\text{int} (B)) - B \subseteq \text{cl} (\text{int} (A)) - A$. Since $A \subseteq B$. As $A$ is strongly $g^*$ closed by the above theorem $\text{cl} (\text{int} (A)) - A$ contains no nonempty closed set, $\text{cl} (\text{int} (B)) - B$ contains no empty closed set. Again by theorem 3.13, $B$ is strongly $g^*$-closed set.

**Theorem 3.8. Theorem 3.16:** Let $A \subseteq Y \subseteq X$ and suppose that $A$ is strongly $g^*$ closed in $X$ then $A$ is strongly $g^*$ closed relative to $Y$.

Proof. Given that $A \subseteq Y \subseteq X$ and $A$ is strongly $g^*$-closed in $X$. To show that $A$ is strongly $g^*$-closed relative to $Y$, let $A \subseteq Y \cap G$, where $G$ is $g$-open in $X$. Since $A$ is strongly $g^*$-closed in $X, A \subseteq G$ implies $\text{cl} (\text{int} (A)) \subseteq G$. (ie) $Y \cap \text{cl} (\text{int} (A)) \subseteq Y \cap G$, where $Y \cap \text{cl} (\text{int} (A))$ is closure of interior of $A$ in $Y$. Thus $A$ is strongly $g^*$-closed relative to $Y$.

**Theorem 3.9.** If a subset $A$ of a topological space $X$ is $gsp$-closed then it is strongly $g^*$-closed but not conversely.

Proof. Suppose that $A$ is $gsp$-closed set in $X$, let $G$ be open set containing $A$. Then $G \supseteq \text{spcl} (A), A \cup G \supseteq A \cup (\text{int} (\text{cl} (\text{int} (A))))$ which implies $G \supseteq \text{int} (\text{cl} (\text{int} (A)))$ as $G$ is open. (ie)$G \supseteq \text{cl} (\text{int} (A)) - A$ is strongly $g^*$-closed set in $X$.

**Example 3.3. Example 3.18:** The converse of the above theorem need not be true from the following example.

Let $X= \{a,b,c\}$ with topology $\tau = \{\Phi, X, \{a\}, \{b,c\}\}$ and $B=\{b\}$. $B$ is not strongly $g^*$ closed. Since $\{b\}$ is a $g$-open set of $(X, \tau)$ such that $B \subseteq \{b\}$ but $\text{cl} (B) = \text{cl} (\{b\}) = \{b, c\} \subseteq \{b\}$. However $B$ is a $gsp$-closed set of $(X, \tau)$.

**Theorem 3.10. Theorem 3.19:** Every $\delta$-closed set is a strongly $g^*$ closed set.

Proof. The Proof of the theorem is immediate from the definition.

**Example 3.4.** The converse of the above theorem need not be true from the following example.

Let $X = \{a, b, c\}, \tau = \{\Phi, X, \{a\}, \{a, b\}\} D = \{a, c\}$. $D$ is not a $\delta$-closed set and also not even closed set. Hence $D$ is strongly $g^*$ closed set.

**Theorem 3.11.** Every $\theta$-closed set is a strongly $g^*$ closed set.

Proof. The Proof of the theorem is immediate from the definition.

**Example 3.5.** The converse of the above theorem need not be true from the following example.

Let $X = \{a, b, c\}, \tau = \{\Phi, X, \{a\}, \{a, b\}\{a, c\}\}$ and $E = \{c\}$. Clearly $E$ is closed and hence strongly $g^*$-closed. $E$ is not $\theta$-closed set of $(X, \tau)$. 


Theorem 3.12. Every strongly $g^*$-closed set in an $\alpha$ g-closed set and hence gs-closed, gsp-closed, gp-closed, gpr closed set and rg closed set but not conversely.

Proof. Let $A$ be a strongly $g^*$-closed set of $(X, \tau)$. By above theorem, $A$ is g-closed. By implications (2.4) in Maki et.al(1993) $A$ is $\alpha$ g-closed. From the investigations of Dontchev(1996) and Ganambal (1997), we know that every g-closed set is gs-closed, gsp-closed, gp-closed, gpr-closed and rg-closed. By above theorem every strongly $g^*$-closed set is gs-closed, gsp closed and rg-closed.

Example 3.6. The converse of the above theorem need not be true from the following example.
Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ $D = \{b\}$. $D$ is not a $\alpha$ g closed, gs closed, gp closed, gpr closed and regular-closed but not strongly $g^*$ closed.

Remark 3.1. The following are the implications of strongly $g^*$-closed set.

References


