On *n* Power Class (*Q*) Operators

S. Panayappan

Department of Mathematics
Government Arts College, Coimbatore – 641018
Tamilnadu, India
panayappan@gmail.com

N. Sivamani

Department of Mathematics
Tamilnadu College of Engineering, Coimbatore- 641659
Tamilnadu, India
sivamanitce@gmail.com

Abstract

In this paper we introduce the new class *n* power class(*Q*) operators acting on a Hilbert space *H*. An operator \( T \in L(H) \) is *n* power class(*Q*) if \( T^{n \ast} T^{2n} = (T^{n} T^{n})^{2} \). We investigate some basic properties of such operator. In general a *n* power class(*Q*) operator need not be a normal operator.

Mathematics Subject Classification: 47B20, 47B99, 47B15.

Keywords: Normal, *n*-Normal, *n* power quasi normal, class(*Q*), Hilbert space.

1 Introduction

Throughout this paper *H* is a Hilbert space and \( L(H) \) is the algebra of all bounded linear operators acting on *H*. An operator \( T \in L(H) \) is called class(*Q*) if \( T^{n \ast} T^{2} = (T^{n \ast})^{2} \), \( T \) is called normal if \( T^{\ast} T = T T^{\ast} \), \( T \) is *n*-normal if \( T^{n \ast} T^{\ast} = T T^{n \ast} \).
$T$ is $n$ power quasi normal if $T^n(T^*T) = (T^*T)^n$ and $T$ is quasi $n$ normal if $T(T^*T^n) = (T^*T^n)T$.

2 Main Results

In this section we investigate some properties of operators in $n$ power class ($Q$).

**Theorem 2.1** If $T \in n$ power class ($Q$) then so are
(i) $kT$ for any real number $k$.
(ii) any $S \in L(H)$ that is unitarily equivalent to $T$.
(iii) the restriction $T_M$ of $T$ to any closed subspace $M$ of $H$ that reduces $T$.

**Proof.** (i) The proof is straightforward.
(ii) Let $S \in L(H)$ be unitarily equivalent to $T$ then there is a unitary operator $U \in L(H)$ such that $S^{2n} = U^*T^{2n}U$ which implies that $S^* = U^*T^*U$.
Thus, $S^*S^{2n} = U^*T^*UU^*T^*US^{2n} = U^*T^*UU^*T^*UU^*T^*U^{2n}U = U^*(T^*)^2T^{2n}U$ and
\[
(S^*S^n)^2 = (U^*T^*UU^*T^*U)^2 = (U^*T^*T^*U)^2 = U^*(T^*T^n)^2U
\]
Since $T^*T^n = (T^*T^n)^2$ we have $S^*S^{2n} = (S^*S^n)^2$.
Thus $S \in n$ power class ($Q$).
(iii) By [2] we have
\[
\left(\frac{T}{M}\right)^2 \left(\frac{T}{M}\right)^{2n} = \left(\frac{T^*}{M}\right) \left(\frac{T^{2n}}{M}\right) = \left(\frac{T^*T^{2n}}{M}\right)
\]
\[
= \left(\frac{T^*T^n}{M}\right)^2 = \left[\left(\frac{T}{M}\right)^2 \left(\frac{T}{M}\right)^{2n}\right]^{\frac{1}{2}}
\]
Thus $T_M \in n$ power class ($Q$).

The following example shows that if unitarily equivalence in theorem 2.1 (ii) is replaced by similarity then the result is need not be true.

**Example 2.2** Consider the two operators $T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acting on the two dimensional Hilbert space then $T \in 2$ power class ($Q$). Now
On n power class \((Q)\) operators

\[
X^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]

and by direct decomposition we show that 
\[XTX^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = S\]
(say). Now again by direct decomposition we show that 
\[
S^* \begin{pmatrix} 64 & -60 \\ -48 & 46 \end{pmatrix}
\]
while 
\[
(S^* S^2)^2 = \begin{pmatrix} 88 & -72 \\ -48 & 40 \end{pmatrix}
\]
Thus \(S\) is similar to \(T\) but \(S \notin 2\ power class(Q)\).

The following example shows that the sum and product of \(2\ power class(Q)\) operators are not \(2\ power class(Q)\).

**Example 2.3** Consider the operators 
\[
S = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}
\]
are \(2\ power class(Q)\) operators on the complex Hilbert space. But \(S + T\) and \(ST\) are not \(2\ power class(Q)\).

**Remark 2.4** If \(T \in n\ power class(Q)\) such that \(T^2 = 0\) then it is not necessarily that \(T = 0\). Consider 
\[
T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
acting on \(R^2\) which is not normal.

**Theorem 2.5** If \(T \in L(H)\) is n-normal then \(T \in n\ power class(Q)\).

**Proof.** Since \(T\) is n-normal then 
\[
T^* T^n = T^n T^*
\]
Pre multiply by \(T^*\) and post multiply by \(T^n\) on both sides we get,
\[
T^* T^n T^* T^n = T^* T^n T^* T^n
\]
\[
T^* T^{2n} = (T^* T^n)^2
\]
Hence \(T \in n\ power class(Q)\).

The following example shows that an operator of \(2\ power class(Q)\) need not be 2 normal.

**Example 2.6** If \(T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\) be an operator acting on three dimensional complex Hilbert space. Then \(T\) is \(2\ power class(Q)\) but it is not 2 normal.
The following examples show that a 2-powerclass($Q$) need not be 3-powerclass($Q$) and vice versa.

**Example 2.7** Consider the operator $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on 2-dimensional complex Hilbert space which is 2-powerclass($Q$) but not 3-powerclass($Q$).

**Example 2.8** Consider the operator $T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ acting on 2-dimensional Hilbert space which is 3-powerclass($Q$) but not 2-powerclass($Q$).

**Theorem 2.9** If $T$ is $n$-powerclass($Q$) and $T$ is quasi $n$ normal then $T$ is $n+1$ powerclass($Q$).

**Proof.** If $T$ is $n$-powerclass($Q$) then $T^{n^2} T^n = (T^* T^n)^2$.

Post multiply by $T^2$ on both sides

$$T^{n^2} T^n T^2 = (T^* T^n)^2 T^2$$

$$T^{n^2} T^{2n+2} = (T^* T^n)^2 T T$$

Since $T$ is quasi $n$ normal we have

$$T^{n^2} T^{2(n+1)} = (T^* T^n)^2 T (T^* T^n)^2 = (T^* T^n)^2$$

Hence $T \in n+1$ powerclass($Q$).

The following example shows that the condition that $T$ is quasi $n$ normal is necessary.

**Example 2.10** Consider the operator $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on $R^2$ which is 2-powerclass($Q$), not quasi $n$ normal and not 3-powerclass($Q$).

**Theorem 2.11** Let $T_1, \ldots, T_m$ be $n$ normal operators in $L(H)$. Then $(T_1 \oplus \ldots \oplus T_m)$ and $(T_1 \otimes \ldots \otimes T_m)$ are $n$ powerclass($Q$) operators.

**Proof.**

$$(T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{n^2} (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{2n}$$

$$= (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{n} (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{n} (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{n} (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^{n}$$

$$= (T_1 ^* \oplus T_2 ^* \oplus \ldots \oplus T_m ^*) (T_1 ^* \oplus T_2 ^* \oplus \ldots \oplus T_m ^*) (T_1 ^n \oplus T_2 ^n \oplus \ldots \oplus T_m ^n) (T_1 ^n \oplus T_2 ^n \oplus \ldots \oplus T_m ^n)$$
On $n$ power class ($Q$) operators

\[ (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (T_1'^n \oplus T_2'^n \oplus \ldots \oplus T_n'^n) (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1'^n \oplus T_2'^n \oplus \ldots \oplus T_n'^n) (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1'^n \oplus T_2'^n \oplus \ldots \oplus T_n'^n) (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1'^n \oplus T_2'^n \oplus \ldots \oplus T_n'^n) (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* \]

Hence $(T_1 \oplus \ldots \oplus T_m)$ is $n$ power class ($Q$).

Now $x_1, x_2, \ldots, x_m \in H$,

\[ (T_1 \oplus T_2 \oplus \ldots \oplus T_m) (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^2^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

\[ = (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^* (T_1 \oplus T_2 \oplus \ldots \oplus T_m)^n (x_1 \oplus x_2 \oplus \ldots \oplus x_m) \]

Hence $(T_1 \oplus \ldots \oplus T_m)$ is $n$ power class ($Q$).

**References**


Received: January, 2012