Inclusion Properties for Certain Subclasses
of Analytic Functions Defined by
a Generalized Multiplier Transformation

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Abstract

Using the principle of subordination, we obtain some inclusion properties of certain subclasses of analytic functions defined by a generalized multiplier transformation. Also inclusion properties of these classes involving the generalized integral operator are obtained.

Mathematical Subject Classification: 30C45

Keywords: Analytic function, multiplier transformation, differential subordination

1. INTRODUCTION

Let $A_p$ denote the class of functions of the form

\begin{equation}
(1.1)\quad f(z) = z^p + \sum_{k=p+1}^\infty a_k z^k, \quad (p \in \mathbb{N} = \{1,2,3\ldots\}).
\end{equation}

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there exists a Schwarz
function $w(z)$, which (by definition) is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalence (See [7],[14] and [15]):

$$f \prec g \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $0 \leq \eta < p$, $p \in \mathbb{N}$, we denote by $S_p^*(\eta), K_p(\eta)$ and $C_p(\eta, \eta)$ the subclasses of $A_p$ consisting of all analytic functions which are, respectively, $p$-valent starlike of order $\eta$, $p$-valent convex of order $\eta$, and $p$-valent close-to-convex functions in $U$ ([17],[19] and [22]).

Let $\Lambda$ be the class of all functions $\phi$ which are analytic and univalent in $U$ and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\text{Re}(\phi(z)) > 0, z \in U$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S_p^*(\eta; \phi), K_p(\eta; \phi)$ and $C_p(\eta, \rho; \phi, \varphi)$ of the class $A_p$, $0 \leq \eta < p$, $0 \leq \rho < p$ and $\phi, \varphi \in \Lambda$, which are defined by:

$$S_p^*(\eta; \phi) = \left\{ f \in A_p : \frac{1}{p-\eta} \left( \frac{zf'(z)}{f(z)} - \eta \right) \prec \phi(z), z \in U \right\},$$

$$K_p(\eta; \phi) = \left\{ f \in A_p : \frac{1}{p-\eta} \left( 1 + \frac{zf''(z)}{f'(z)} - \eta \right) \prec \phi(z), z \in U \right\},$$

and

$$C_p(\eta, \rho; \phi, \varphi) = \left\{ f \in A_p : \frac{1}{p-\rho} \left( \frac{zf'(z)}{g(z)} - \rho \right) \prec \varphi(z), z \in U, \text{where}, g \in S_p^*(\eta; \phi) \right\}.$$

From these definitions, we can obtain some well known subclasses of $A_p$, by special choices of the functions $\phi$ and $\varphi$. For example, we have

$$S_p^*\left( \eta; \frac{1+z}{1-z} \right) = S_p^*(\eta), \ K_p\left( \eta; \frac{1+z}{1-z} \right) = K_p(\eta) \text{ and } C_p\left( 0,0; \frac{1+z}{1-z}, \frac{1+z}{1-z} \right) = C_p.$$

Now we define the new generalized multiplier transformation $T^n_{p,\alpha,\beta}$ on $A_p$ as below:
**Definition 1.1.** Let \( p \in N, \ n \in N_0 = N \cup \{0\}, \ \beta \geq 0 \) and \( \alpha \) a real number with \( \alpha + p\beta > 0 \). Then for \( f \in A_p \), we define the operator \( I_{p,a,\beta}^n \) by

\[
I_{p,a,\beta}^0 f(z) = f(z),
\]

\[
I_{p,a,\beta}^1 f(z) = \frac{\alpha f(z) + \beta f'(z)}{\alpha + p\beta},
\]

\[
\ldots,
\]

\[
I_{p,a,\beta}^n f(z) = I_{p,a,\beta}(I_{p,a,\beta}^{n-1} f(z)).
\]

**Remark 1.2.** We observe that \( I_{p,a,\beta}^n : A_p \to A_p \), is a linear operator and for \( f(z) \) given by (1.1), we have

\[
I_{p,a,\beta}^n f(z) = \sum_{k=0}^{n} \left( \frac{\alpha \beta k}{\alpha + p\beta} \right)^n a_k z^k.
\]

It follows from (1.2) that

\[
I_{p,a,\beta}^0 f(z) = f(z),
\]

\[
(\alpha + p\beta)I_{p,a,\beta}^{n+1} f(z) = \alpha I_{p,a,\beta}^n f(z) + \beta I_{p,a,\beta}^{n} f(z), \beta > 0,
\]

and

\[
I_{p,a,\beta}^n (I_{p,a,\beta}^{n_2} f(z) = I_{p,a,\beta}^{n_2} (I_{p,a,\beta}^n f(z), \text{ for all } n_1, n_2 \in N_0.
\]

We note that

- \( I_{a,\beta}^n f(z) = I_{a,\beta}^n f(z) \) (See [24]).
- \( I_{p,a,\beta}^n f(z) = I_p^\alpha (\alpha) f(z), \alpha > -p \) (See [1], [21] and [23]).
- \( I_{p,\alpha^2,\beta}^n f(z) = I_p^n (\beta, l) f(z), l > -p, \beta \geq 0 \) (See Catas [8]).
- \( I_{p,0,\beta}^n f(z) = D_p^n f(z) \) (See [4], [12] and [16]).

**Remark 1.3.** a) i) \( I_p^n (\alpha) f(z) \) was considered in [1], [21] and [23] for \( \alpha \geq 0 \) and \( I_p^n (\beta, l) f(z) \) was defined in [8] for \( l \geq 0, \beta \geq 0 \). ii) \( I_p^n (l) f(z) = I_p^n (1, l) f(z), l > -p \). So our results in this paper are improvement of corresponding results proved earlier for \( I_p^n (\alpha) f(z) \) or \( I_p^n (\beta, l) f(z) \) to \( \alpha > -p \) or \( -p > l, \) respectively.
Next, by using the operator $I_{p,a,\beta}^n$, we introduce the following subclasses of analytic functions for $\phi, \varphi \in \Lambda$, $p \in N, n \in N_0, \beta \geq 0, \alpha$ a real number with $\alpha + p\beta > 0, 0 \leq \eta < p$ and $0 \leq \rho < p$.

\[
S_{p,a,\beta}^n(\eta; \phi) = \left\{ f \in A_p : I_{p,a,\beta}^n f(z) \in S_p^n(\eta; \phi) \right\},
K_{p,a,\beta}^n(\eta; \phi) = \left\{ f \in A_p : I_{p,a,\beta}^n f(z) \in K_p(\eta; \phi) \right\},
\]
and
\[
C_{p,a,\beta}^n(\eta, \rho; \phi, \varphi) = \left\{ f \in A_p : I_{p,a,\beta}^n f(z) \in C_p(\eta, \rho; \phi, \varphi) \right\}.
\]

We also note that
\[
f(z) \in K_{p,a,\beta}^n(\eta; \phi) \iff \frac{zf'(z)}{p} \in S_{p,a,\beta}^n(\eta; \phi).
\]

In particular, we set
\[
S_{p,a,\beta}^n \left( \eta; \frac{1 + Az}{1 + Bz} \right) = S_{p,a,\beta}^n(\eta; A, B), -1 \leq B < A \leq 1,
\]
and
\[
K_{p,a,\beta}^n \left( \eta; \frac{1 + Az}{1 + Bz} \right) = K_{p,a,\beta}^n(\eta; A, B), -1 \leq B < A \leq 1.
\]

In section 2, some preliminary results are mentioned. In section 3, we show that $S_{p,a,\beta}^{n+1}(\eta; \phi) \subset S_{p,a,\beta}^n(\eta; \phi), K_{p,a,\beta}^{n+1}(\eta; \phi) \subset K_{p,a,\beta}^n(\eta; \phi)$ and $C_{p,a,\beta}^{n+1}(\eta, \rho; \phi, \varphi) \subset C_{p,a,\beta}^n(\eta, \rho; \phi, \varphi)$. In section 4, we study inclusion properties of classes $S_{p,a,\beta}^n(\eta; \phi), K_{p,a,\beta}^n(\eta; \phi)$ and $C_{p,a,\beta}^n(\eta, \rho; \phi, \varphi)$, involving generalized Libera integral operator.

2. PRELIMINARY LEMMAS

The following lemmas will be required in our investigation.
Lemma 2.1 ([11]). Let $\phi$ be convex, univalent in $U$ with $\phi(0) = 1$ and $\text{Re}(\kappa \phi(z) + \gamma) > 0$, $\kappa, \gamma \in C$. If $p(z)$ is analytic in $U$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \gamma} \prec \phi(z), (z \in U) \implies p(z) \prec \phi(z), (z \in U).$$

Lemma 2.2 ([15]). Let $\phi$ be convex, univalent in $U$ and $w$ be analytic in $U$ with $\text{Re}(w(z)) \geq 0$. If $p(z)$ is analytic in $U$ with $p(0) = \phi(0)$, then

$$p(z) + w(z)zp'(z) \prec \phi(z), (z \in U) \implies p(z) \prec \phi(z), (z \in U).$$

3. INCLUSION PROPERTIES INVOLVING THE OPERATOR $I_{p,a,\beta}^n$.

Unless otherwise mentioned we shall assume that $\beta \geq 0, \alpha$ a real number with $\alpha + p\beta > 0$, $p \in N$, $n \in N_0$, $0 \leq \eta < p$ and $0 \leq \rho < p$, throughout this paper.

Theorem 3.1. Let $f \in A_p$ and let $\phi \in \Lambda$ with $\text{Re}((p - \eta)\phi(z) + \eta + (\alpha / \beta)) > 0$. Then

$$S_{p,a,\beta}^{n+1}(\eta;\phi) \subset S_{p,a,\beta}^n(\eta;\phi).$$

Proof. Let $f(z) \in S_{p,a,\beta}^{n+1}(\eta;\phi)$ and set

$$(3.1) \quad p(z) = \frac{1}{p - \eta} \left( \frac{z(I_{p,a,\beta}^n f(z))'}{I_{p,a,\beta}^n f(z)} - \eta \right),$$

Where $p(z)$ is analytic in $U$ with $p(0) = 1$. Using (1.4) in (3.1), we get

$$(3.2) \quad \left( \frac{\alpha + p\beta}{\beta} \right) I_{p,a,\beta}^n f(z) = (p - \eta)p(z) + \eta + (\alpha / \beta).$$

Differentiating (3.2) logarithmically with respect to $z$, we obtain

$$(3.3) \quad \frac{1}{p - \eta} \left( \frac{z(I_{p,a,\beta}^{n+1} f(z))'}{I_{p,a,\beta}^{n+1} f(z)} - \eta \right) = p(z) + \frac{zp'(z)}{(p - \eta)p(z) + \eta + (\alpha / \beta)}, z \in U.$$

Applying Lemma 2.1 to (3.3), it follows that $p \prec \phi$, i.e. $f \in S_{p,a,\beta}^n(\eta;\phi)$.

Theorem 3.2. Let $f \in A_p$ and let $\phi \in \Lambda$ with $\text{Re}((p - \eta)\phi(z) + \eta + (\alpha / \beta)) > 0$. Then

$$K_{p,a,\beta}^{n+1}(\eta;\phi) \subset K_{p,a,\beta}^n(\eta;\phi).$$
Proof. Applying (1.5) and Theorem 3.1, we conclude that
\[ f \in K_{p,a,\beta}^{n+1}(\eta;\phi) \Rightarrow \frac{zf^n}{p} \in S_{p,a,\beta}^{n+1}(\eta;\phi) \subset S_{p,a,\beta}^n(\eta;\phi) \]
\[ \Leftrightarrow \frac{zf^n}{p} \in S_{p,a,\beta}^n(\eta;\phi) \]
\[ \Leftrightarrow f \in K_{p,a,\beta}^n(\eta;\phi). \]

Taking \( \phi(z) = \frac{1 + Az}{1 + Bz} \), \(-1 \leq B < A \leq 1\), \(z \in U\), in Theorem 3.1 and Theorem 3.2, we have the following corollary.

**Corollary 3.3.** Let \( f \in A_p \). Then \( S_{p,a,\beta}^{n+1}(\eta;A,B) \subset S_{p,a,\beta}^n(\eta;A,B) \) and \( K_{p,a,\beta}^{n+1}(\eta,A,B) \subset K_{p,a,\beta}^n(\eta,A,B) \).

By using Lemma 2.2, we obtain the following inclusion relation for the class \( C_{p,a,\beta}^n(\eta,\rho;\phi,\varphi) \).

**Theorem 3.4.** Let \( f \in A_p \) and let \( \phi,\varphi \in \Lambda \) with \( \text{Re}((p - \eta)\phi(z) + \eta + (\alpha/\beta)) > 0 \). Then
\[ C_{p,a,\beta}^{n+1}(\eta,\rho;\phi,\varphi) \subset C_{p,a,\beta}^n(\eta,\rho;\phi,\varphi). \]
Proof. Let \( f \in C_{p,a,\beta}^{n+1}(\eta,\rho;\phi,\varphi) \), then by definition there exists a function \( g \in S_{p,a,\beta}^{n+1}(\eta;\phi) \) such that
\[ \frac{1}{p - \rho} \left( \frac{z(I_{p,a,\beta}^{n+1}f(z))}{I_{p,a,\beta}^n g(z)} - \rho \right) < \varphi(z), z \in U. \]

Now, let \( p(z) = \frac{1}{p - \rho} \left( \frac{z(I_{p,a,\beta}^{n+1}f(z))}{I_{p,a,\beta}^n g(z)} - \rho \right) \), where \( p(z) \) is analytic in \( U \) with \( p(0) = 1 \).

Using (1.4), we have
\[ (3.4) \quad \left( \frac{\alpha + p\beta}{\beta} \right) I_{p,a,\beta}^{n+1} f(z) = (\alpha/\beta) I_{p,a,\beta}^n f(z) + [(p - \rho)p(z) + \rho] I_{p,a,\beta}^n g(z). \]

Differentiating (3.4) with respect to \( z \) and multiplying by \( z \), we get
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(3.5) \( \left( \frac{\alpha + p\beta}{\beta} \right) z(I_{p,a,\beta}^{n+1}f(z))^{'} = \left( \frac{\alpha}{\beta} \right) z(I_{p,a,\beta}^{n}f(z))^{'} + [(p - \rho)p(z) + \rho]z(I_{p,a,\beta}^{n}g(z))^{'} + (p - \rho)zp'(z)(I_{p,a,\beta}^{n}g(z)). \)

Since \( g \in S_{p,a,\beta}^n(\eta;\phi) \), then by Theorem 3.1, we have \( g \in S_{p,a,\beta}^n(\eta;\phi) \). Let

\[ h(z) = \frac{1}{p - \eta} \left( \frac{z(I_{p,a,\beta}^{n}g(z))}{I_{p,a,\beta}^{n}g(z)} - \eta \right). \]

Applying (1.4) again, we get

(3.6) \( \left( \frac{\alpha + p\beta}{\beta} \right) \frac{I_{p,a,\beta}^{n+1}g(z)}{I_{p,a,\beta}^{n}g(z)} = (p - \eta)h(z) + \eta + (\alpha / \beta). \)

From (3.5) and (3.6), we have

\[ \frac{1}{p - \rho} \left( \frac{z(I_{p,a,\beta}^{n+1}f(z))^{'} - \rho}{I_{p,a,\beta}^{n}g(z)} \right) = p(z) + \frac{zp'(z)}{(p - \eta)h(z) + \eta + (\alpha / \beta)}, \text{ } z \in U. \]

Since \( 0 \leq \eta < p \) and \( h(z) < \phi(z) \) in \( U \), then \( \text{Re}((p - \eta)h(z) + \eta + (\alpha / \beta)) > 0 \). So by taking \( w(z) = 1/[((p - \eta)h(z) + \eta + (\alpha / \beta))] \) and applying Lemma 2.2, we can show that \( p < \phi \), so that \( f \in C_{p,a,\beta}^n(\eta,\rho;\phi,\phi) \), which proves Theorem 3.4.

4. INCLUSION PROPERTIES INVOLVING THE INTEGRAL OPERATOR \( F_{c} \)

In this section we consider the generalized Libera integral operator \( F_{p,c} \) (See [6], [13] and [18]), defined by

(4.1) \[ F_{p,c}(f)(z) = \frac{c + p}{z^c} \int_{0}^{t} f(t)dt, (c > -p, f \in A_p). \]

Theorem 4.1. Let \( c > -p \) and let \( \phi \in \Lambda \) with \( \text{Re}((p - \eta)\phi(z) + \eta + c) > 0 \). If \( f \in S_{p,a,\beta}^n(\eta;\phi) \), then \( F_{p,c}(f) \in S_{p,a,\beta}^n(\eta;\phi) \).

Proof. Let \( f \in S_{p,a,\beta}^n(\eta;\phi) \) and set...
\[ p(z) = \frac{1}{p-\eta} \left( \frac{z(I_{p,a,\beta}^n f_{p,\alpha}(f)(z))'}{I_{p,a,\beta}^n f_{p,\alpha}(f)(z)} - \eta \right), \]

Where \( p \) is analytic in \( U \) with \( p(0) = 1 \). From (4.1), we have
\[ z(I_{p,a,\beta}^n f_{p,\alpha}(f)(z))' = (c+1)I_{p,a,\beta}^n f_{p,\alpha}(f)(z) - cI_{p,a,\beta}^n f_{p,\alpha}(f)(z). \]

Using (4.3) in (4.2), we get
\[ (c+p)\frac{I_{p,a,\beta}^n f_{p,\alpha}(f)(z)}{I_{p,a,\beta}^n f_{p,\alpha}(f)(z)} = (p-\eta) p(z) + \eta + c. \]

Differentiating (4.4) logarithmically with respect to \( z \), we obtain
\[ \frac{1}{p-\eta} \left( \frac{z(I_{p,a,\beta}^n f_{p,\alpha}(f)(z))'}{I_{p,a,\beta}^n f_{p,\alpha}(f)(z)} - \eta \right) = p(z) + \frac{zp'(z)}{(p-\eta)p(z)+\eta+c}. \]

Applying Lemma 2.1 to (4.5), we conclude that \( F_{p,\alpha}(f)(z) \in S_{p,a,\beta}^n(\eta;\phi) \).

Similarly applying (1.5) and Theorem 4.1, we have the following result.

**Theorem 4.2.** Let \( c > -p \) and let \( \phi \in \Lambda \) with \( \text{Re}((p-\eta)\phi(z) + \eta + c) > 0 \). If \( f \in K_{p,a,\beta}^n(\eta;\phi) \), then \( F_{p,\alpha}(f) \in K_{p,a,\beta}^n(\eta;\phi) \).

From Theorem 4.1 and Theorem 4.2, we have the following corollary:

**Corollary 4.3.** Let \( f \in A_p \) and \( c > -p \). If \( f \in S_{p,a,\beta}^n(\eta;A,B) \) (or \( K_{p,a,\beta}^n(\eta;A,B) \)), then \( F_{p,\alpha}(f) \in S_{p,a,\beta}^n(\eta;A,B) \) (or \( K_{p,a,\beta}^n(\eta;A,B) \)).

**Theorem 4.4.** Let \( c > -p \) and let \( \phi,\varphi \in \Lambda \) with \( \text{Re}((p-\eta)\phi(z) + \eta + c) > 0 \). If \( f \in C_{p,a,\beta}^n(\eta,\rho;\phi,\varphi) \), then \( F_{p,\alpha}(f) \in C_{p,a,\beta}^n(\eta,\rho;\phi,\varphi) \).

**Proof.** Let \( f \in C_{p,a,\beta}^n(\eta,\rho;\phi,\varphi) \). Then there exists a function \( g \in S_{p,a,\beta}^n(\eta;\phi) \) such that
\[ \frac{1}{p-\rho} \left( \frac{z(I_{p,a,\beta}^n f_{p,\alpha}(f)(z))'}{I_{p,a,\beta}^n f_{p,\alpha}(f)(z)} - \rho \right) < \varphi(z), \quad z \in U. \]

We set
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\[ p(z) = \frac{1}{p - \rho} \left( z(I_{p,a,\beta}^n F_{p,c}(f)(z))' - \rho \right) \]

where \( p \) is analytic in \( U \) with \( p(0) = 1 \). Since \( g \in S_{p,a,\beta}^n(\eta; \phi) \), we have from Theorem 4.1, that \( F_{p,c}(g) \in S_{p,a,\beta}^n(\eta; \phi) \). Using (4.3) we obtain

\[ [(p - \rho)p(z) + \rho]I_{p,a,\beta}^n F_{p,c}(g)(z) + cI_{p,a,\beta}^n F_{p,c}(f)(z) = (c + p)I_{p,a,\beta}^n f(z). \]

Then by simple calculations, we get

\[ (c + p) \frac{z(I_{p,a,\beta}^n f(z))'}{I_{p,a,\beta}^n F_{p,c}(g)(z)} = [(p - \rho)p(z) + \rho][(p - \eta)h(z) + \eta + c] + (p - \rho)zp'(z), \]

where, \( h(z) = \frac{1}{p - \eta} \left( z(I_{p,a,\beta}^n F_{p,c}(g)(z))' - \eta \right) \). Hence, we have

\[ \frac{1}{p - \rho} \left( z(I_{p,a,\beta}^n f(z))' - \rho \right) = p(z) + \frac{zp'(z)}{(p - \eta)h(z) + \eta + c}. \]

The remaining part of the proof is similar to that of Theorem 3.4 and so we omit it.

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Received: January, 2012