k-Weyl Fractional Integral

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Abstract

In this paper we present a generalization to the k-calculus of the Weyl Fractional Integral. Show the semigroup property and calculate their Fractional Fourier Transform.

Mathematics Subject Classification: 26A33, 42A38

Keywords: k-Fractional Calculus. Weyl fractional integral

I Introduction and Preliminaries

In [1] Diaz and Pariguan has defined new functions called the k-Gamma function and the Pochhammer k-symbol that are generalization of the classical Gamma function and the classical Pochhammer symbol. Later in 2012, Mubeen and Habibullah [11] has introduced the k-fractional integral of the Riemann-Liouville type. The purpose of this paper is introduce a fractional integral operator of Weyl type and study that may be posible to exprese this integral operator as certain convolution with the singular kernel of Riemann-Liouville.

The classical Weyl integral operator of order $\alpha$ is defined (cf. [10]) in the form

$$W^\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_u^\infty (t - u)^{\alpha-1} f(t)dt \quad (I.1)$$

$u \geq 0$, $\alpha > 0$ and $f$ a function belonging to $S(\mathbb{R})$ the Schwartzian space of functions, and $\Gamma(z)$ is the Gamma Euler function given by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$
\[ \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt, \quad z \in \mathbb{C}, \quad \text{Re}(z) > 0, \quad \text{cf. [6]} \quad (I.2) \]

In order to simplify the study of the integral operators it is convenient to treat as the convolution of the function \( f \) with the singular Riemann-Liouville that is given: for \( \alpha > 1 \), and \( t > 0 \), \( j_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \).

It is therefore necessary to remember the definitions of two kinds of convolutions given in the following definitions.

**Definition 1** Let \( f \) and \( g \) be functions belonging to \( L^1(\mathbb{R}^+) \), the usual or classical convolution product is given by

\[ (f \ast g)(t) = \int_0^t f(x-t)g(x)dx, \quad t > 0 \quad (I.3) \]

**Definition 2** Let \( f \) and \( g \) be function belonging to \( L^1(\mathbb{R}^+) \). Miana (cf. [9]) introduce the convolution product \( \circ \) as the integral

\[ (f \circ g)(t) = \int_t^\infty f(x-t)g(x)dx, \quad t \geq 0 \quad (I.4) \]

Thus, the Weyl fractional integral (I.1) can be regarded as the convolution

\[ W^\alpha f(x) = \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \circ f \right)(x) \quad (I.5) \]

To develop the final part of our work is also needed.

**Definition 3** **Fractional Fourier Transform** Let \( f \) be a function belonging to \( \phi(\mathbb{R}) \), the Lizorkin space.

The Fractional Fourier transform (FFT) of order \( \alpha \), \( 0 < \alpha \leq 1 \), is defined as (cf. [13])

\[ \hat{f}_\alpha(\omega) = \mathfrak{F}_\alpha[f](\omega) = \int_\mathbb{R} e^{i\omega^{1/\alpha}t}f(t)dt \quad (I.6) \]

When if \( \alpha = 1 \), (I.5) reduces at the classical Fourier Transform given by the following integral

\[ \hat{f}(\omega) = \mathfrak{F}[f](\omega) = \int_\mathbb{R} f(t)e^{i\omega t}dt \quad (I.7) \]
II k-Weyl fractional Integral

In order to generalize the Weyl fractional integral we need to introduce an analogue to the Riemann-Liouville singular kernel. Then there is the following

**Definition 4** Let $\alpha$ be a real number, $0 < \alpha < 1$, $k > 0$. The $k$-Riemann-Liouville singular kernel is given by

$$j_{\alpha,k}(t) = \frac{t^{\frac{\alpha}{k} - 1}}{k \Gamma_k(\alpha)} \quad t > 0 \quad (\text{II.1})$$

where $\Gamma_k(z)$ is the $k$-generalization of the Gamma Euler function defined by

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t}{k}} dt, \quad z \in \mathbb{C}, \ Re(z) > 0 \ (\text{cf. [1]}) \quad (\text{II.2})$$

**Definition 5** Let $\alpha$ be a real number, $0 < \alpha < 1$, $k > 0$. The $k$-Weyl Fractional Integral is defined as

$$W_k^\alpha f(x) = j_{\alpha,k}(t) \ast f(t) = \frac{1}{k \Gamma_k(\alpha)} \int_x^\infty (t - x)^{\frac{\alpha}{k} - 1} f(t) dt \quad (\text{II.3})$$

where $\ast$ denote the convolution due to Miana (cf. [9]).

We next prove that the $W_k^\alpha f$ operator verified the semigroup property. Previously we prove the

**Lemma 1** Let $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < 1$, $0 < \beta < 1$ and $k > 0$ then

$$j_{\alpha,k} \ast j_{\beta,k} = j_{\alpha+\beta,k} \quad (\text{II.4})$$

where $\ast$ denote the classical convolution.

**Proof.** By definition (II.1) we have

$$j_{\alpha,k}(t) \ast j_{\beta,k}(t) = \frac{t^{\frac{\alpha}{k} - 1}}{k \Gamma_k(\alpha)} \ast \frac{t^{\frac{\beta}{k} - 1}}{k \Gamma_k(\beta)} \quad (\text{II.5})$$

calling $\mu = \frac{\alpha}{k}$ and $\omega = \frac{\beta}{k}$, it result

$$j_{\alpha,k}(t) \ast j_{\beta,k}(t) = \frac{t^{\frac{\mu}{k} - 1}}{k \Gamma_k(\mu)} \ast \frac{t^{\frac{\omega}{k} - 1}}{k \Gamma_k(\omega)} =$$
\[ \frac{\mu^{-1}}{k\Gamma_k(k\mu)} * \frac{\omega^{-1}}{k\Gamma_k(k\omega)} \]  

(II.6)

using the relation between Gamma function and k-Gamma function 
\[ \Gamma_k(x) = k^x \Gamma(x) \], cf. [1], we have

\[ j_{\alpha,k}(t) * j_{\beta,k}(t) = \frac{1}{k^2} \frac{1}{k^{\mu + \omega - 2}} \frac{\mu^{-1}}{\Gamma(\mu)} * \frac{\omega^{-1}}{\Gamma(\omega)} = \]

\[ \frac{1}{k} \frac{1}{k^{\mu + \omega - 1}} \frac{\mu + \omega - 1}{\Gamma(\mu + \omega)} = \frac{1}{k} \frac{\mu + \omega - 1}{\Gamma_k(k(\mu + \omega))} = \]

\[ \frac{1}{k} \frac{t_{\alpha + \beta}^{-1}}{\Gamma_k(\alpha + \beta)} = j_{\alpha + \beta,k}(t) \]

**Lemma 2** Let \( \alpha, \beta \in \mathbb{R}, 0 < \alpha < 1, 0 < \beta < 1 \) and \( k > 0 \) then

\[ W_k^\alpha \left[ W_k^\beta f \right] = W_k^{\alpha + \beta} f = W_k^\beta \left[ W_k^\alpha f \right] \]  

(II.7)

**Proof.** By definition (II.3) we have

\[ W_k^\alpha \left[ W_k^\beta f \right] = j_{\alpha,k} \circ [j_{\beta,k} \circ f] \]

taking into account the lemma 2 and the relation \( f \circ (g \circ h) = (f \circ g) \circ h \), cf. [9] it result

\[ W_k^\alpha \left[ W_k^\beta f \right] = (j_{\alpha,k} * j_{\beta,k}) \circ f = j_{\alpha + \beta,k} \circ f = W_k^{\alpha + \beta} f \]

**Lemma 3** Let \( \alpha, \beta \in \mathbb{R}, 0 < \alpha < 1, 0 < \beta < 1 \) and \( k > 0 \). The Fourier Transform of k-Riemann-Liouville singular kernel in the point \( -\omega^{1/\beta} \) is

\[ \mathcal{F} \left[ j_{\alpha,k} \right] (-\omega^{1/\beta}) = \frac{1}{k^{\alpha/k}} |\omega|^{-\alpha/k} \left[ \cos \left( \frac{\alpha \pi}{2k} \right) - i \text{ sign}(\omega^{1/\beta}) \text{ sen} \left( \frac{\alpha \pi}{2k} \right) \right] \]

(II.8)
Proof. By definition (II.1) and Fourier Transform (I.7) we have
\[ \mathfrak{F}[j_{a,k}](\omega) = \frac{1}{k \Gamma_k(\alpha)} \int_{\mathbb{R}} e^{i(-\omega^{1/\beta})t} t^{\frac{\beta}{k} - 1} dt = \]
\[ \frac{1}{k \Gamma_k(\alpha)} \int_0^{+\infty} e^{i(-\omega^{1/\beta})t} t^{\frac{\beta}{k} - 1} dt = \]
\[ \frac{1}{k \Gamma_k(\alpha)} \left[ \int_0^{+\infty} \cos(\omega^{1/\beta} t) t^{\frac{\beta}{k} - 1} dt - i \int_0^{+\infty} \sin(\omega^{1/\beta} t) t^{\frac{\beta}{k} - 1} dt \right] \quad \text{(II.9)} \]

Applying formulae 3.761.9 from [2] we have
\[ \int_0^{+\infty} \cos(\omega^{1/\beta} t) t^{\frac{\beta}{k} - 1} dt = \Gamma\left( \frac{\alpha}{k} \right) \left| \omega^{1/\beta} \right|^{-\frac{\alpha}{k}} \cos\left( \frac{\alpha \pi}{2k} \right) \quad \text{(II.10)} \]

and by formulae 3.761.4 from [2] we have
\[ \int_0^{+\infty} \sin(\omega^{1/\beta} t) t^{\frac{\beta}{k} - 1} dt = \Gamma\left( \frac{\alpha}{k} \right) \left| \omega^{1/\beta} \right|^{-\frac{\alpha}{k}} \sin\left( \frac{\alpha \pi}{2k} \right) \text{sign}(\omega^{1/\beta}) \quad \text{(II.11)} \]

replacing (II.10) and (II.11) in (II.9) and using the relation between Gamma and k-Gamma function it result
\[ \mathfrak{F}[j_{a,k}](\omega) = \frac{1}{k \Gamma_k(\alpha)} \Gamma\left( \frac{\alpha}{k} \right) \left| \omega^{1/\beta} \right|^{-\frac{\alpha}{k}} \left[ \cos\left( \frac{\alpha \pi}{2k} \right) + i \text{sign}(\omega^{1/\beta}) \sin\left( \frac{\alpha \pi}{2k} \right) \right] = \]
\[ \frac{k^{1-\frac{\alpha}{k}}}{k \Gamma_k(\alpha)} \Gamma_k(\alpha) \left| \omega \right|^{-\frac{\alpha}{k}} \left[ \cos\left( \frac{\alpha \pi}{2k} \right) + i \text{sign}(\omega^{1/\beta}) \sin\left( \frac{\alpha \pi}{2k} \right) \right] = \]
\[ \frac{1}{k^{\frac{\beta}{k}}} \left| \omega \right|^{-\frac{\alpha}{k}} \left[ \cos\left( \frac{\alpha \pi}{2k} \right) + i \text{sign}(\omega^{1/\beta}) \sin\left( \frac{\alpha \pi}{2k} \right) \right] \]

\[ \Box \]

Lemma 4 Let \( \alpha, \beta \in \mathbb{R}, 0 < \alpha < 1, 0 < \beta < 1 \) and \( k > 0 \). The Fractional Fourier Transform of order \( \beta \) of the k-Weyl fractional integral is given by
\[ \mathfrak{F}_\beta [W_k^\alpha f](\omega) = \frac{\left| \omega \right|^{-\frac{\alpha}{k}}}{k^{\frac{\beta}{k}}} \left[ \cos\left( \frac{\alpha \pi}{2k} \right) + i \text{sign}(\omega^{1/\beta}) \sin\left( \frac{\alpha \pi}{2k} \right) \right] \mathfrak{F}_\beta [f](\omega) \quad \text{(II.12)} \]
Proof. By definition (II.3) we have

\[ \mathcal{F}_\beta [W_k^\alpha f] (\omega) = \mathcal{F}_\beta [j_{\alpha,k} \circ f] (\omega) \]  

(II.13)

taking into account the convolution theorem of the Fractional Fourier Transform (cf. [13]) it result

\[ \mathcal{F}_\beta [W_k^\alpha f] (\omega) = \mathcal{F}_\beta [j_{\alpha,k} \circ f] (\omega) = \mathcal{F}_\beta [-\omega^{1/\beta}] \mathcal{F}_\beta [f] (\omega) = \]

\[ \left| \omega \right|^{-\frac{\alpha}{k}} \left[ \cos \left( \frac{\alpha \pi}{2k} \right) + i \text{sign}(\omega^{1/\beta}) \text{sen} \left( \frac{\alpha \pi}{2k} \right) \right] \mathcal{F}_\beta [f] (\omega) \]

Example of application

Let \( \nu \in \mathbb{R}, 0 < \nu \leq 1 \) and \( f(t) = (t + a)^{-\mu} \) the potential function. We will calculate the k-Weyl Fractional Integral of \( f(t) \). In fact

\[ W_k^\nu(t + a)^{-\mu} = \frac{1}{k \Gamma_k(\nu)} \int_0^{\infty} (t - x)^{\frac{\nu}{k} - 1} (t + a)^{-\mu} dt \]  

(II.14)

making the change of variables

\[ \xi = \frac{t - x}{t + a}, \quad dt = \frac{x + a}{(1 - \xi)^2} d\xi \]

we have

\[ W_k^\nu(t + a)^{-\mu} = \frac{(x + a)^{\frac{k}{k} - \mu}}{\Gamma_k(\nu)} 1 \int_0^1 \xi^{\frac{k}{k} - 1} (1 - \xi)^{\nu - \frac{k}{k} - 1} d\xi = \]

\[ \frac{(x + a)^{\frac{k}{k} - \mu}}{\Gamma_k(\nu)} B_k(\nu, \mu k - \nu) \]  

(II.15)

where \( B_k(z, \omega) \) is the k-Beta function (cf. [1]).

Then

\[ W_k^\nu(t + a)^{-\mu} = \frac{(x + a)^{\frac{k}{k} - \mu}}{\Gamma_k(\nu)} B_k(\nu, \mu k - \nu) = \frac{(x + a)^{\frac{k}{k} - \mu} \Gamma_k(\nu) \Gamma_k(\mu k - \nu)}{\Gamma_k(\nu) \Gamma_k(k \mu)} = \]

\[ (x + a)^{\frac{k}{k} - \mu} \frac{\Gamma_k(\mu k - \nu)}{\Gamma_k(k \mu)} \]
References


Received: January, 2012