Global Strong Solutions of the Time-Dependent Ginzburg-Landau Model for Superconductivity with a New Gauge

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Abstract

We consider the initial boundary value problem of the time-dependent Ginzburg-Landau equations in superconductivity. We introduce a new gauge, which has been used by Tao [13] to study wave maps. We prove the global existence of strong solutions of the problem with this new gauge.

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1 Introduction

We shall study the following Ginzburg-Landau equations in superconductivity:

\[ \eta \psi_t + i \eta k \phi \psi + \left( \frac{i}{k} \nabla + A \right)^2 \psi + (|\psi|^2 - 1)\psi = 0, \quad (1.1.1) \]

\[ A_t + \nabla \phi + \text{curl}^2 A + \text{Re} \left\{ \left( \frac{i}{k} \nabla \psi + \psi A \right) \bar{\psi} \right\} = \text{curl} H, \quad (1.1.2) \]
in \( QT := (0, T) \times \Omega \) with boundary and initial conditions

\[
\begin{align*}
\nabla\psi \cdot \nu &= 0, \quad A \cdot \nu = 0, \quad \text{curl} \times \nu = H \times \nu \quad \text{on} \quad (0, T) \times \partial\Omega, \\
(\psi, A)(x, 0) &= (\psi_0, A_0)(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^3.
\end{align*}
\]

Here the unknowns \( \psi, A, \) and \( \phi \) are \( \mathbb{C} \)-valued, \( \mathbb{R}^d \)-valued, and \( \mathbb{R} \)-valued functions respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. \( H := H(x) \) is the applied magnetic field, \( \eta \) and \( k \) are Ginzburg-Landau positive constants, and \( i := \sqrt{-1} \). \( \Omega \) is a bounded domain with smooth boundary \( \partial\Omega \) and \( \nu \) is the outward normal to \( \partial\Omega \). \( \overline{\psi} \) denotes the complex conjugate of \( \psi \), \( \text{Re} \psi := (\psi + \overline{\psi})/2 \), \( |\psi|^2 := \psi \overline{\psi} \) is the density of superconducting carriers. \( T \) is any given positive constant.

It is well known that the Ginzburg-Landau equations are gauge invariant, that is, if \( (\psi, A, \phi) \) is a solution of (1.1.1)-(1.1.4), then for any real-valued smooth function \( \chi \), \((\psi e^{ik\chi}, A + \nabla \chi, \phi - \chi_t)\) is also a solution of (1.1.1)-(1.1.4). So in order to obtain the well-posedness of the problem, we need to impose gauge condition. One usually has three types of the gauge condition:

- **Coulomb gauge:** \( \text{div} \ A = 0 \) in \( \Omega \) and \( \int_{\Omega} \phi dx = 0 \).
- **Lorentz gauge:** \( \phi = -\text{div} \ A \) in \( \Omega \).
- **Temporal gauge:** \( \phi = 0 \) in \( \Omega \).

In this note we will present a new Lorentz gauge:

\[
\phi_t = -\text{div} \ A \quad \text{in} \quad \Omega. \tag{1.1.5}
\]

For the initial data \( \psi_0 \in H^1(\Omega), |\psi_0| \leq 1 \) in \( \Omega \) and \( A_0 \in H^1(\Omega) \), Chen, Elliott, Tang [1], Chen, Hoffmann, Liang [3], Du [4], and Tang [11] proved the existence and uniqueness of global strong solutions to (1.1.1)-(1.1.4) in the case of the Coulomb and Lorentz as well as temporal gauges. For the initial data \( \psi_0 \in H^1(\Omega), A_0 \in H^1(\Omega) \), Tang and Wang [12] obtained the existence and uniqueness of global strong solutions. Fan and Jiang [6] showed the existence of global weak solutions when \( \psi_0, A_0 \in L^2(\Omega) \). Fan and Gao [5], Fan and Ozawa [7, 8, 9] proved some uniqueness criteria of weak solutions. Zaouch [14] proved the existence of time-periodic solutions when the applied magnetic field \( H \) is time periodic. Phillips and Shin [10], Chen and Hoffmann [2] studied the well-posedness of classical solutions to the nonisothermal models for superconductivity. The aim of this paper is to prove a similar result with our new gauge (1.1.5). We will prove
Theorem 1.1. Let $\psi_0 \in H^1(\Omega), |\psi_0| \leq 1$ in $\Omega$, $A_0 \in H^1(\Omega), \phi_0 \in H^1(\Omega)$ and $H \in H^1(\Omega)$. Then there exists a unique strong solution $(\psi, A, \phi)$ of (1.1.1)-(1.1.4) in the case of the Lorentz gauge (1.1.5), such that

$$
|\psi| \leq 1 \text{ in } Q_T, \psi \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \psi_t \in L^2(0, T; L^2),
A \in L^\infty(0, T; H^1), A_t \in L^2(0, T; L^2),
\phi \in L^\infty(0, T; H^1), \phi_t \in L^\infty(0, T; L^2)
$$

for any $T > 0$.

Remark 1.1. The Lorentz gauge (1.1.5) has been used in the study of wave maps [13].

Remark 1.2. With the choice of new gauge (1.1.5), we have new equation (2.2.4) (see below) for $\phi$ which is hyperbolic. The most difficult part of the proof is to deal with the right hand side of (2.2.4). To overcome the difficulty, we rewrite the right hand side of (2.2.4) by using the equation (1.1.1) and derive the energy estimate of $\phi$ through the Ginzburg-Landau free energy.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Since it is easy to prove that the problem (1.1.1)-(1.1.5) has a unique local strong solution and thus we omit the details here. We only need to prove a priori estimates which ensure (1.1.6).

To begin with, it has been proved in [1, 3, 4, 11] that

$$
|\psi| \leq 1 \text{ in } Q_T.
$$

(2.2.1)

by the maximum principle.

It is well-known that the Ginzburg-Landau free energy given by

$$
G(\psi, A) := \frac{1}{2} \int_\Omega \left( \left| \frac{i}{k} \nabla \psi + \psi A \right|^2 + \frac{1}{2} (|\psi|^2 - 1)^2 + |\text{curl} A - H|^2 \right) dx,
$$

satisfies

$$
\frac{dG}{dt} = -\int_\Omega \left( \eta |\psi_t + ik\phi\psi|^2 + |A_t + \nabla \phi|^2 \right) dx \leq 0.
$$

(2.2.2)

On the other hand, it follows from (1.1.1) that

$$
\text{Re div} \left\{ \left( \frac{i}{k} \nabla \psi + \psi A \right) \overline{\psi} \right\} = \text{Re} \left\{ \eta ki(\psi_t + ik\phi\psi) \overline{\psi} \right\}.
$$

(2.2.3)
Taking $\text{div}$ to (1.1.2) and using (1.1.5), we see that
\[
-\phi_{tt} + \Delta \phi = -\text{div} \left\{ \text{Re} \left( \frac{i}{k} \nabla \psi + \psi A \right) \overline{\psi} \right\}. \tag{2.2.4}
\]

It follows from (1.1.2) and (1.1.3) that
\[
\nabla \phi \cdot \nu = 0 \text{ on } (0, T) \times \partial \Omega. \tag{2.2.5}
\]

Testing (2.2.4) by $-\phi_t$ and using (2.2.2), (2.2.3), (2.2.1) and (2.2.5), we get
\[
\frac{1}{2} \frac{d}{dt} \int (\phi_t^2 + |\nabla \phi|^2) \, dx \leq \int \eta k |\psi_t + i k \phi \psi| |\phi_t| \, dx \\
\leq C \int (\phi_t^2 + |\psi_t + i k \phi \psi|^2) \, dx,
\]
which gives
\[
\sup_t \int_\Omega (\phi_t^2 + |\nabla \phi|^2) \, dx \leq C, \quad \|\text{div} \, A\|_{L^\infty(0,T;L^2)} \leq C. \tag{2.2.6} \tag{2.2.7}
\]

It follows from (2.2.2) that
\[
\|\text{curl} \, A\|_{L^\infty(0,T;L^2)} \leq C. \tag{2.2.8}
\]

Using the well-known Poincaré inequality:
\[
\|A\|_{L^2} \leq C(\|\text{div} \, A\|_{L^2} + \|\text{curl} \, A\|_{L^2}),
\]
we have
\[
\|A\|_{L^\infty(0,T;H^1)} \leq C. \tag{2.2.9}
\]

By (2.2.1), (2.2.2) and (2.2.9), we obtain
\[
\|\psi\|_{L^\infty(0,T;H^1)} \leq C. \tag{2.2.10}
\]

Since
\[
\int |A_t + \nabla \phi|^2 \, dx = \int (|A_t|^2 + |\nabla \phi|^2 + 2 \phi \phi_t) \, dx,
\]
it follows from (2.2.2) that
\[
\|\phi\|_{L^\infty(0,T;L^2)} \leq C, \quad \|A_t\|_{L^2(0,T;L^2)} \leq C. \tag{2.2.11} \tag{2.2.12}
\]
By (2.2.1), (2.2.2) and (2.2.11), we obtain
\[ \|\psi\|_{L^2(0,T;L^2)} \leq C. \] (2.2.13)

Finally, testing (1.1.1) by \(-\Delta \overline{\psi}\), taking the real part, and using (2.2.1), (2.2.2), (2.2.9) and (2.2.10), we arrive at
\[ \|\psi\|_{L^2(0,T;H^2)} \leq C. \]

This completes the proof. \(\square\)

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References


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