Second Order Conditions for Constrained Optimization Problems in Binormed Spaces

Lahrech Samir
Mohamed first university
Faculty of science, Oujda, Morocco
MATSI Laboratory
lahrechsamir@yahoo.fr

Bouaicha Amal
Mohamed first university
Faculty of science, Oujda, Morocco
MATSI Laboratory

Abstract

We already know the benefit of metric regularity in optimization. This leads to a number of important ways of recognizing extremum points. But, when we deal with constrained optimization problems in normed spaces, this tool in general cannot be used to derive optimality conditions as we shall see in the example given in this note. The main purpose of this paper, is to present prototypical second order conditions for constrained optimization in binormed spaces. Our approach is based on a recent extension of Fréchet differentiability (approach of Taylor mappings) and the notion of $b$-metric regularity with respect to the pair of norms which is equivalent to the notion of metric regularity in normed spaces.

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1 Introduction to $b$-metric regularity and differentiability with respect to the pair of norms

We shall be working in binormed space $(E, \|\cdot\|_1, \|\cdot\|_2)$ such that $(E, \|\cdot\|_2)$ is a Banach space and for some $c > 0 \|\cdot\|_1 \leq c\|\cdot\|_2$.

1.1 Differentiability with respect to the pair of norms

In this paragraph, we briefly review some basic notions and results related to differentiability in binormed spaces, which will be used later.

Let $U$ be an open set of $(E, \|\cdot\|_1)$, $h$ a map acting from $U$ into a Banach space $(Y, \|\cdot\|)$, and let $\bar{x} \in U$.

According to , $h$ is said to be $(\|\cdot\|_1, \|\cdot\|_2)$-Fréchet differentiable at a point $\bar{x}$ if there exists a continuous linear operator from $(E, \|\cdot\|_2)$ to $(Y, \|\cdot\|)$ denoted $\nabla h(\bar{x})$ such that

$$h(\bar{x} + v) = h(\bar{x}) + \nabla h(\bar{x})v + \varepsilon(v)\|v\|_2,$$

where $\varepsilon(v) \to 0$ in $(Y, \|\cdot\|)$ as $v \to 0$ with respect to the norm $\|\cdot\|_1$.

**Remark 1.1** Let us remark that if $h$ is $\|\cdot\|_1$-Fréchet differentiable at a point $\bar{x}$, then $h$ is also $(\|\cdot\|_1, \|\cdot\|_2)$-Fréchet differentiable at $\bar{x}$ and if $h$ is $(\|\cdot\|_1, \|\cdot\|_2)$-Fréchet differentiable at $\bar{x}$, then $h$ is $\|\cdot\|_2$-Fréchet differentiable at $\bar{x}$.

If $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|_2$, then $h$ is $(\|\cdot\|_1, \|\cdot\|_2)$-Fréchet differentiable at $\bar{x}$ iff $h$ is $\|\cdot\|_1$-Fréchet differentiable at $\bar{x}$.

According to , $h$ is said to be $(\|\cdot\|_1, \|\cdot\|_2)$ twice Fréchet differentiable at $\bar{x}$ if there exists a continuous linear operator denoted $\nabla h(\bar{x}) \in \mathcal{L}((E, \|\cdot\|_2), (Y, \|\cdot\|))$ and a bilinear continuous mapping denoted $\nabla^2 h(\bar{x}) \in \mathcal{B}((E, \|\cdot\|_2), (Y, \|\cdot\|))$ such that:

$$h(\bar{x} + v) = h(\bar{x}) + \nabla h(\bar{x})v + \frac{1}{2}\nabla^2 h(\bar{x})(v,v) + \|v\|_2^2 \varepsilon(v),$$

where $\varepsilon(v) \to 0$ in $(Y, \|\cdot\|)$ as $v \to 0$ with respect to the norm $\|\cdot\|_1$.

Notice that in general, even if $h$ is $(\|\cdot\|_1, \|\cdot\|_2)$ twice Fréchet differentiable at $\bar{x}$, $h$ is not necessarily $(\|\cdot\|_1, \|\cdot\|_2)$ Fréchet differentiable at $\bar{x}$.

**Remark 1.2** If $h$ is $\|\cdot\|_1$-twice Fréchet differentiable at a point $\bar{x}$, then $h$ is also $(\|\cdot\|_1, \|\cdot\|_2)$-twice Fréchet differentiable at $\bar{x}$ and if $h$ is $(\|\cdot\|_1, \|\cdot\|_2)$-twice Fréchet differentiable at $\bar{x}$, then $h$ is $\|\cdot\|_2$-twice Fréchet differentiable at $\bar{x}$.

If $\|\cdot\|_1$ is equivalent to the norm $\|\cdot\|_2$, then $h$ is $(\|\cdot\|_1, \|\cdot\|_2)$-twice Fréchet differentiable at $\bar{x}$ iff $h$ is $\|\cdot\|_1$-twice Fréchet differentiable at $\bar{x}$.
1.2  \( b \)-metric regularity with respect to the pair of norms

Throughout this section we consider an open set \( U \) of \( (E, \| \cdot \|_1) \), a subset \( S \) of \( U \), an Euclidean space \( (Y, \| \cdot \|) \), and a mapping \( h: U \to Y \). Let \( \bar{x} \in S \).

Let \( d_S^1 \) denote the distance function to the set \( S \) with respect to the norm \( \| \cdot \|_1 \).

In this section, we will introduce the notion of \( b \)-metric regularity in binormed spaces which generalize the notion of metric regularity in normed spaces.

The potential advantage of \( b \)-metric regularity over concept of metric regularity might be related to the fact that it offers a rich calculus and applications as we will see in the example given at the end of the paper.

**Definition 1.3** we say that \( h \) is \( b \)-regular on \( S \) at \( \bar{x} \) with respect to the pair of norms \( (\| \cdot \|_1, \| \cdot \|_2) \) if

\[
\forall t_n \to +\infty, \forall x_n \in S \cap h^{-1}(h(\bar{x})) \text{ such that } t_n(x_n - \bar{x}) \text{ converge in } (E, \| \cdot \|_1) \text{ (}t_n(x_n - \bar{x}) \text{) is bounded in } (E, \| \cdot \|_2).
\]

**Example 1.4** Put \( S = \{\bar{x}, x_1, x_2\} \), where \( \bar{x}, x_1, x_2 \in E \) and let \( h: E \to Y \) be any function satisfying: \( h(\bar{x}) = 0 \), \( h(x_i) \neq 0 \) for \( i = 1, 2 \). Then \( h \) is \( b \)-regular on \( S \) at \( \bar{x} \) with respect to the pair of norms \( (\| \cdot \|_1, \| \cdot \|_2) \).

Let us remark that if \( \| \cdot \|_1 \) is equivalent to the norm \( \| \cdot \|_2 \), then every function \( h: E \to Y \) is \( b \)-regular on \( S \) at \( \bar{x} \) with respect to the pair of norms \( (\| \cdot \|_1, \| \cdot \|_2) \).

**Definition 1.5** \( h \) is said to be co-metrically regular on \( S \) at \( \bar{x} \) with respect to the norm \( \| \cdot \|_1 \) if there exists \( k > 0 \) such that

\[
d_{S \cap h^{-1}(h(\bar{x}))}^1(y) \leq k\|h(z) - y\|
\]

\( \forall y \to h(\bar{x}) \) in \( (Y, \| \cdot \|) \) and \( \forall z \to \bar{x} \) in \( (\text{conv } S, \| \cdot \|_1) \), where \( \text{conv } S \) denote the closed convex hull of \( S \) with respect to the norm \( \| \cdot \|_1 \).

**Definition 1.6** We shall say that \( h \) is \( b \)-metrically regular on \( S \) at \( \bar{x} \) with respect to the pair of norms \( (\| \cdot \|_1, \| \cdot \|_2) \) if it is co-metrically regular on \( S \) at \( \bar{x} \) with respect to the norm \( \| \cdot \|_1 \) and \( b \)-regular on \( S \) at \( \bar{x} \) with respect to the pair of norms \( (\| \cdot \|_1, \| \cdot \|_2) \).

**Remark 1.7** Let us remark that if \( \| \cdot \|_1 \) is equivalent to the norm \( \| \cdot \|_2 \) and \( S \) is a convex closed set of \( (E, \| \cdot \|_1) \), then \( h \) is \( b \)-metrically regular on \( S \) at \( \bar{x} \) with respect to the pair of norms \( (\| \cdot \|_1, \| \cdot \|_2) \) iff \( h \) is metrically regular on \( S \) at \( \bar{x} \) with respect to the norm \( \| \cdot \|_1 \).

With this notion of \( b \)-metric regularity we are now ready for the main result of this paper, which gives second order conditions for constrained optimization.
2 Second order optimality conditions

In this section we present prototypical second order conditions for constrained optimization in binormed spaces. Our approach is simple and elegant blend of convex analysis, differentiability in binormed spaces and \( \beta \)-metric regularity.

Let \((E, ||.||_1, ||.||_2)\) be a binormed space such that \((E, ||.||_2)\) is a Banach space and \(||.||_2\) is finer than \(||.||_1\).

Consider an open set \(U\) of \((E, ||.||_1)\), a Euclidean space \((Y, ||.||)\), a subset \(S\) of \(U\). Given any function \(h : U \rightarrow Y\) that is \(||.||_2\)-twice Fréchet differentiable at \(\bar{x} \in S\). We consider the nonlinear optimization problem

\[
(P) \quad \begin{cases} 
\inf f(x), \\
h(x) = 0, \ x \in \text{conv } S.
\end{cases}
\]

where the objective function \(f : U \rightarrow R\) is assumed to be \(||.||_1, ||.||_2\)-twice Fréchet differentiable at \(\bar{x}\).

Define the narrow critical cone at \(\bar{x}\) by:

\[
C(\bar{x}) = \{d \in R_+(\text{conv } S - \bar{x}) | \nabla f(\bar{x})d \leq 0, \nabla h(\bar{x})d = 0\}.
\]

Let \(N^2_{\text{conv } S}(\bar{x})\) denote the normal cone to \(\text{conv } S\) at \(\bar{x}\) with respect to the norm \(||.||_2\) in the sense of convex analysis.

Before proving the main result of our paper, let us recall the well known result which gives second order conditions for a point to be a local minimum for the problem \((P)\) with respect to the norm \(||.||_1\).

**Theorem 2.1** Assume that \(S\) is convex and closed in \((E, ||.||_1)\), the functions \(f : U \rightarrow R\), \(h : U \rightarrow Y\) are \(||.||_1\)-twice Fréchet differentiable at \(\bar{x}\) and \(h\) is metrically regular on \(S\) at \(\bar{x}\) with respect to the norm \(||.||_1\). Suppose also that the point \(\bar{x}\) is a local minimum for the problem \((P)\) with respect to the norm \(||.||_1\), that the direction \(d\) lies in the narrow critical cone \(C(\bar{x})\), and that the conditions

\[
-\nabla^2 h(\bar{x})(d,d) \in \text{core} \nabla h(\bar{x})(R_+(S - \bar{x}))
\]

\[
\nabla h(\bar{x})z = -\nabla^2 h(\bar{x})(d,d) \text{ for some } z \in R_+(S - \bar{x}) \text{ hold.}
\]

Then there exists a multiplier \(\lambda\) in \(Y\) such that the lagrangian

\[
L(.) = f(.) + (\lambda, h(.))
\]

satisfies the conditions

\[-\nabla L(\bar{x}) \in N^2_S(\bar{x})\]

and

\[
\nabla^2 L(\bar{x})(d,d) \geq 0.
\]
This test can be richly refined using the notion of differentiability in binormed spaces and the notion of $b$-metric regularity.

We can now extend the result given in Theorem 2.1, in the case where $f$ and $h$ are not necessarily $\|\|_1$-twice Fréchet differentiable at $\bar{x}$.

**Theorem 2.2** Assume that the function $f : U \to \mathbb{R}$ is $(\|\|_1, \|\|_2)$-twice Fréchet differentiable at $\bar{x}$, $\nabla f(\bar{x}) \in \mathcal{L}(\mathcal{E}, \|\|_1, \mathbb{R})$ and $\nabla^2 f(\bar{x}) \in B(\mathcal{E}, \|\|_1, \mathbb{R})$. Let $h : U \to Y$ be a $\|\|_2$-twice Fréchet differentiable mapping at $\bar{x}$. Suppose also that $h$ is $b$-metrically regular on $S$ at $\bar{x}$ with respect to the pair of norms $(\|\|_1, \|\|_2)$ and $\bar{x}$ is a local minimum for the problem $(\mathcal{P})$ with respect to the norm $\|\|_1$. Let $d$ be any direction of the narrow critical cone $C(\bar{x})$ such that:

$$-\nabla^2 h(\bar{x})(d, d) \in \text{core}[\nabla h(\bar{x})(\mathbb{R}^+ (\text{conv } S - \bar{x})]$$

and

$$\nabla h(\bar{x})z = -\nabla^2 h(\bar{x})(d, d) \text{ for some } z \in \mathbb{R}^+ (\text{conv } S - \bar{x}).$$

Then there exists a multiplier $\lambda$ in $Y$ such that the Lagrangian

$$L(.) = f(.) + (\lambda, h(.))$$

satisfies the conditions

$$-\nabla L(\bar{x}) \in N_{\text{conv } S}^2(\bar{x})$$

and

$$\nabla^2 L(\bar{x})(d, d) \geq 0.$$ 

**Remark 2.3** Let us remark first full that we cannot use the classical Theorem 2.1 to prove our result since $f$ and $h$ are not necessarily $\|\|_1$-twice Fréchet differentiable at $\bar{x}$, under our hypothesis. Notice also that to derive our result, it is not possible to apply the same theorem by considering $\|\|_2$ instead of $\|\|_1$, since under our hypothesis again, $h$ is not necessarily metrically regular on $S$ at $\bar{x}$ with respect to the norm $\|\|_2$.

If $S$ is convex and closed in $(\mathcal{E}, \|\|_1)$ and if $\|\|_2$ is equivalent to the norm $\|\|_1$, then by Remark 1.7, we recapture the result given in Theorem 2.1. So, our Theorem 2.2 extend and generalizes the result given in Theorem 2.1.

Let us prove now the Theorem 2.2.

**Proof.**

Consider first the convex problem

$$(Q) \quad \left\{ \begin{array}{l} \inf \nabla f(\bar{x})z, \\
\nabla h(\bar{x})z = -\nabla^2 h(\bar{x})(d, d), \\
z \in \mathbb{R}^+ (\text{conv } S - \bar{x}). \end{array} \right.$$
Suppose the point $z$ is feasible for problem (Q). It is clear that, for small real $t \geq 0$, the path

$$x(t) = \bar{x} + td + \frac{t^2}{2}z$$

lies in $\text{conv} \ S$. Moreover, the quadratic approximation shows this path almost satisfies the original constraint for small $t$:

$$h(x(t)) = h(\bar{x}) + t\nabla h(\bar{x})d + \frac{t^2}{2}\nabla^2 h(\bar{x})z + \frac{\nabla^2 h(\bar{x})}{2}[td + \frac{t^2}{2}z, td + \frac{t^2}{2}z] + t^2\varepsilon_1(t),$$

where $\varepsilon_1(t) \to 0$ in $(Y, \|\cdot\|)$. On the other hand, $h$ is co-metrically regular on $S$ at $\bar{x}$ with respect to the norm $\|\cdot\|_1$. Therefore, there is a real constant $k > 0$ such that, for small $t \geq 0$, we have

$$d^1_{\text{Sh}}(0)(x(t)) < k\|h(x(t))\| = o(t^2).$$

Thus we can perturb the path slightly to obtain a set of points

$$\{\bar{x}(t) | t \geq 0\} \subset S \cap h^{-1}(0)$$

satisfying $\|\bar{x}(t) - x(t)\|_1 \leq k\|h(x(t))\| = o(t^2)$. Consequently, there exists $\varepsilon_2(t)$ tending to 0 in $(E, \|\cdot\|_1)$ such that $\bar{x}(t) = x(t) + t^2\varepsilon_2(t)$.

Since $\bar{x}$ is a $\|\cdot\|_1$-local minimizer for the problem (P), then

$$f(\bar{x}) \leq f(\bar{x}(t)).$$

On the other hand,

$$f(\bar{x}(t)) = f(\bar{x} + td + \frac{t^2}{2}z + t^2\varepsilon_2(t)) = f(\bar{x}) + \nabla f(\bar{x})(td + \frac{t^2}{2}z + t^2\varepsilon_2(t))$$

$$+ \frac{\nabla^2 f(\bar{x})}{2}[td + \frac{t^2}{2}z + t^2\varepsilon_2(t), td + \frac{t^2}{2}z + t^2\varepsilon_2(t)] + \varepsilon_3(t)\|td + \frac{t^2}{2}z + t^2\varepsilon_2(t)\|_2^2,$$

where $\varepsilon_3(t) \to 0$ in $(R, \|\cdot\|)$. Consequently, using the fact that $\nabla f(\bar{x}) \in \mathcal{L}((E, \|\cdot\|_1), R)$ and $\nabla^2 f(\bar{x}) \in \mathcal{B}((E, \|\cdot\|_1), R)$, we deduce

$$0 \leq t\nabla f(\bar{x})d + \frac{t^2}{2}(\nabla f(\bar{x})z + \nabla^2 f(\bar{x})(d, d)) + \varepsilon_3(t)\|td + \frac{t^2}{2}z + t^2\varepsilon_2(t)\|_2^2 + o(t^2).$$

Let $(t_n)$ be any positive sequence tending to 0 in $R$. Put $y_n = \bar{x}(t_n)$. Then, for $n$ large enough, we have

$$0 \leq t_n\nabla f(\bar{x})d + \frac{t_n^2}{2}(\nabla f(\bar{x})z + \nabla^2 f(\bar{x})(d, d)) + \varepsilon_3(t_n)t_n^2\frac{y_n - \bar{x}}{t_n}\|_2^2 + o(t_n^2).$$
Since $h$ is $b$-regular on $S$ at $\bar{x}$ with respect to the pair of norms $(\|\cdot\|_1,\|\cdot\|_2)$, $y_n \in S \cap h^{-1}(0)$ and $\frac{y_n - \bar{x}}{n}$ converges to $d$ in $(E,\|\cdot\|_1)$, it follows $(\frac{y_n - \bar{x}}{n})$ is bounded in $(E,\|\cdot\|_2)$. Hence, $\nabla f(\bar{x})d \geq 0$, so in fact $\nabla f(\bar{x})d = 0$, since $d$ lies in $C(\bar{x})$. We deduce $\nabla f(\bar{x})z + \nabla^2 f(\bar{x})(d,d) \geq 0$.

We have therefore shown the optimal value of the convex problem ($\mathcal{Q}$) is at least $-\nabla^2 f(\bar{x})(d,d)$. For the final step in the proof, we rewrite problem $\mathcal{P}$ in Fenchel form:

$$\inf_{z \in E} \{ \langle \nabla f(\bar{x}), z \rangle + \delta_{\{R_+ (\text{conv} S - \bar{x})\}}(z) + \delta_{\{-\nabla^2 h(\bar{x})(d,d)\}}(\nabla h(\bar{x})z) \}.$$ 

Since $-\nabla^2 h(\bar{x})(d,d) \in \text{core}[\nabla h(\bar{x})(R_+ (\text{conv} S - \bar{x})])$, we can apply Fenchel duality in the topological dual $(E,\|\cdot\|_2)'$ to deduce there exists $\lambda \in Y$ satisfying

$$-\nabla^2 f(\bar{x})(d,d) \leq -\delta_{\text{conv} S}(\nabla h(\bar{x})^* \lambda - \nabla f(\bar{x})) + \langle \lambda, \nabla^2 h(\bar{x})(d,d) \rangle >,$$

whence the result.

**Example 2.4** Let $\Omega$ a bounded domain in $\mathbb{R}^2$, $E = W^{1,2}_0(\Omega)$ the Sobolev space with the usual norm $\|\cdot\|_2 = \|\|_W^{1,2}(\Omega)$. Let also $p$ and $\varepsilon$ such that $0 < \varepsilon < 1$, $\varepsilon + 2 < p < \infty$. Set $\|\cdot\|_1 = \|\|_{L^p(\Omega)}$. Remark that $(E,\|\cdot\|_2)$ is a Banach separable space.

Since $W^{1,2}_0(\Omega) \hookrightarrow L^p(\Omega)$ then $(E,\|\cdot\|_1,\|\cdot\|_2)$ is a binormed space such that for some $c > 0 \|\cdot\|_1 \leq c\|\cdot\|_2$.

Set $g(u) = \|u\|^{\varepsilon+2}$ and consider the functional $G$ defined on $E$ by

$$G(x) = \int_{\Omega} g(x(s))ds.$$

Then $G$ is $(\|\cdot\|_1,\|\cdot\|_2)$-twice Fréchet differentiable at every $x \in E$ and

$$G^{(1)}(x)h = \int_{\Omega} g'(x(s))h(s)ds,$$

$$G^{(2)}(x)(h_1, h_2) = \int_{\Omega} g''(x(s))h_1(s)h_2(s)ds.$$

Moreover, $\nabla G(x) \in \mathcal{L}(E,\|\cdot\|_1, R)$ and $\nabla^2 G(x) \in \mathcal{B}(E,\|\cdot\|_1, R)$. Let $\bar{x} \in E$ and put $S = \{\bar{x}, x_1, x_2\}$, where $x_1, x_2 \in E$. Let $h : E \to Y$ be any $\|\cdot\|_2$-twice Fréchet differentiable mapping at $\bar{x}$ satisfying: $h(\bar{x}) = 0$, $h(x_i) \neq 0$ for $i = 1, 2$ and $h$ is co-metrically regular on $S$ at $\bar{x}$ with respect to the norm $\|\cdot\|_1$. Then $h$ is $b$-regular on $S$ at $x$ with respect to the pair of norms $(\|\cdot\|_1,\|\cdot\|_2)$. Consequently, $h$ is $b$-metrically regular on $S$ at $\bar{x}$ with respect to the pair of norms $(\|\cdot\|_1,\|\cdot\|_2)$.

We assert that $G$ is $(\|\cdot\|_1,\|\cdot\|_2)$-strictly differentiable at every $x \in E$. Indeed,
suppose by the contrary that there exists \( x \in E, \varepsilon > 0, x'_m \in E, h_m \in E \) such that \( x'_m \to x \) in \( (E, \| \cdot \|_1) \), \( h_m \to 0 \) in \( (E, \| \cdot \|_1) \) and \( |r(x'_m, h_m)| > \varepsilon \|h_m\|_2 \), where

\[
    r(x', h) = G(x' + h) - G(x') - G^{(1)}(x)h.
\]

Then,

\[
    |r(x'_m, h_m)| = \left| \int_{\tau_0}^{\tau_1} [g'(x'_m(s) + th_m(s)) - g'(x(s))]h_m(s)\,dt \right|.
\]

Let \( p' > 1 \) such that \( \frac{1}{p} + \frac{1}{p'} = 1 \). Since \( p > \varepsilon + 2 \) then \( p > p'(\varepsilon + 1) \). Thus, \( x'_m \to x \) in \( (E, \| \cdot \|_{L^{p'}(\varepsilon + 1)}(\Omega)) \) and \( h_m \to 0 \) in \( (E, \| \cdot \|_{L^{p'}(\varepsilon + 1)}(\Omega)) \). Without loss of generality we can suppose that there exists \( Z_1 \in L^{p'}(\varepsilon + 1)(\Omega) \) such that almost everywhere in \( \Omega \)

\[
    |x'_m(s)| + |h_m(s)| \leq Z_1(s), \quad h_m(s) \to 0, \quad x'_m(s) \to x(s).
\]

Using the Holder inequality, we deduce that

\[
    |r(x'_m, h_m)| \leq \left[ \int_{\tau_0}^{\tau_1} |g'(x'_m(s) + th_m(s)) - g'(x(s))|^{p'} dt ds \right]^{\frac{1}{p'}} \|h_m\|_{L^p(\Omega)}.
\]

Consequently,

\[
    |r(x'_m, h_m)| \leq c \left[ \int_{\tau_0}^{\tau_1} |g'(x'_m(s) + th_m(s)) - g'(x(s))|^{p'} dt ds \right]^{\frac{1}{p'}} \|h_m\|_2.
\]

But this contradicts the fact that \( |r(x'_m, h_m)| > \varepsilon \|h_m\|_2 \) since

\[
    \int_{\tau_0}^{\tau_1} |g'(x'_m(s) + th_m(s)) - g'(x(s))|^{p'} dt ds \to 0,
\]

by the dominated convergence theorem.

Let us remark that \( G \) is not \( \| \cdot \|_1 \)-Fréchet differentiable at any point \( x \in E \). Indeed, let \( \alpha_m \to \infty \) and \( d_m \to \infty \) such that \( |g(d_m)| \geq \alpha_m |d_m|^p \). By the countable additivity of the Lebesgue measure

\[
    \exists C > 0, \exists \Omega' \subset \Omega \text{ such that } \mu(\Omega') > 0, \text{ dist}(\Omega', \partial \Omega) > 0 \text{ and } \forall s \in \Omega' \mid x(s) \mid \leq C.
\]

In this case, put \( D = \max \{ g(u) : |u| \leq C \} < \infty \). Choose \( \Omega_m \subset \Omega' \) such that \( \mu(\Omega_m) = |d_m|^{-\frac{p}{2}} |\alpha_m|^{-\frac{1}{2}} \) for large \( m \).

Let \( h_m \) defined by

\[
    h_m(s) = \begin{cases} 
        d_m - x(s), & s \in \Omega_m, \\
        0, & s \in \Omega \setminus \Omega_m. 
    \end{cases}
\]

It follows then that \( \|h_m\|_1 \to 0 \) and

\[
    |G(x + h_m) - G(x)| \geq \alpha_m |d_m|^p \mu(\Omega_m) - D\mu(\Omega_m) = |\alpha_m|^{\frac{1}{2}} - D\mu(\Omega_m) \to +\infty.
\]
Let \( k_m \in C_0^\infty(\Omega) \) such that \( \|k_m - h_m\|_1 \to 0 \). Then \( \|k_m\|_1 \to 0 \), but \( G(x+k_m) - G(x) \to \infty \) since the Lebesgue integral is absolutely continuous. Therefore, \( G \) is not \( \|\cdot\|_1 \)-Fréchet differentiable at \( x \).
Let us remark first full that we cannot use the classical Theorem 2.1 to derive second order conditions for \( \bar{x} \) to be a local minimum for the problem

\[
(P) \quad \begin{cases} 
\inf G(x) \\
\quad h(x) = 0, \\
\quad x \in \text{conv } S,
\end{cases}
\]

since \( G \) is not \( \|\cdot\|_1 \)-twice Fréchet differentiable at \( \bar{x} \). But using our test we can easily derive second order conditions for \( \bar{x} \) to be a local minimum for the problem \( (P) \) since all the hypothesis of the test are satisfied.

References


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