Fixed Point Theorems for Generalized Contractive Mappings on Cone Metric Spaces

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Abstract

In this paper, some new fixed point theorems for generalized contractive type mappings in cone metric spaces are established.

Mathematics Subject Classification: 47H10, 54H25

Keywords: Fixed point, Generalized contractive mapping, Cone metric space

1 Introduction and preliminaries

Recently, the authors [6] proved a fixed point theorem for mappings satisfying cone integral type contractive condition in cone metric spaces.

In this paper, we establish some new generalized contractive type conditions for mappings defined on cone metric spaces and prove some new fixed point theorems for these mappings. Our results are generalizations of results in [2, 3, 4, 5].

Let \((E, \tau)\) be a topological vector space and \(P \subset E\). Then, \(P\) is called a \textit{cone} whenever

\(\text{i) } P\) is closed, non-empty and \(P \neq \{0\}\);
\(\text{ii) } ax + by \in P\) for all \(x, y \in P\) and non-negative real numbers \(a, b\);
\(\text{iii) } P \cap (-P) = \{0\}\).

Given a cone \(P \subset E\), we define a partial ordering \(\leq\) with respect to \(P\) by

\[ x \leq y \text{ if and only if } y - x \in P. \]

We write \(x < y\) to indicate that \(x \leq y\) but \(x \neq y\).
For \( x, y \in P \), \( x \ll y \) stand for \( y - x \in \text{int}(P) \), where \( \text{int}(P) \) is the interior of \( P \).

For a nonempty set \( X \), a mapping \( d : X \times X \to E \) is called cone metric \([3]\) on \( X \) if the following conditions are satisfied:

(i) \( 0 \leq d(x, y) \) for all \( x, y \in X \), and \( d(x, y) = 0 \) if and only if \( x = y \);
(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(iii) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

From now on, we assume that \( E \) is a normed space, \( P \) is a cone in \( E \) with \( \text{int}(P) \neq \emptyset \) and \( \leq \) is a partial ordering with respect to \( P \). And let \( X \) be a cone metric space with a cone metric \( d \).

The following definitions are in \([3]\).

Let \( x \in X \), and let \( \{x_n\} \) be a sequence of points of \( X \). Then

(1) \( \{x_n\} \) converges to \( x \) (denoted by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \)) if for any \( c \in \text{int}(P) \), there exists \( N \) such that for all \( n > N \), \( d(x_n, x) \ll c \).

(2) \( \{x_n\} \) is Cauchy if for any \( c \in \text{int}(P) \), there exists \( N \) such that for all \( n, m > N \), \( d(x_n, x_m) \ll c \).

(3) \( (X, d) \) is complete if every Cauchy sequence in \( X \) is convergent.

A function \( F : P \to P \) is called \( \ll \)-increasing if, for each \( x, y \in P \), \( x \ll y \) if and only if \( f(x) \ll f(y) \).

Let \( F : P \to P \) be a function such that

(\( F1 \)) \( F(t) = 0 \) if and only if \( t = 0 \);

(\( F2 \)) \( F \) is \( \ll \)-increasing;

(\( F3 \)) \( F \) is surjective.

We denote by \( \mathcal{S}(P, P) \) the family of functions satisfying (\( F1 \)), (\( F2 \)) and (\( F3 \)).

**Example 1.1.** (1) Let \( F(t) = t \) for each \( t \in P \). Then \( F \in \mathcal{S}(P, P) \).

(2) Let \( a, b \in E \), and let \( [a, b] = \{x \in E : x = tb + (1 - t)a, t \in [0, 1]\} \).

Suppose that a mapping \( \varphi : P \to P \) is non-vanishing cone integrable on each \( [a, b] \subset P \) such that, for each \( c \in \text{int}(P) \), \( \int_0^c \varphi d_P \in \text{int}(P) \) (see \([6]\)).

Let \( F(t) = \int_0^t \varphi d_P \). Then \( F \in \mathcal{S}(P, P) \).

Note that if \( \varphi : P \to P \) is subadditive, then \( F \) is subadditive (see \([6]\)).

**Lemma 1.2.** \([1]\) Let \( E \) be a topological vector space. If \( c_n \in E \) and \( c_n \to 0 \), then for each \( c \in \text{int}(P) \) there exists \( N \) such that \( c_n \ll c \) for all \( n > N \).

2 Fixed point theorems

**Theorem 2.1.** Let \( (X, d) \) be a complete cone metric space. Suppose that a mapping \( T : X \to X \) satisfies
for all \( x, y \in X \), where \( k \in [0, \frac{1}{2}) \) and \( F \in \mathcal{S}(P, P) \) such that
(1) \( F \) is subadditive;
(2) if, for \( \{c_n\} \subset P \), \( \lim_{n \to \infty} F(c_n) = 0 \) then \( \lim_{n \to \infty} c_n = 0 \).

Then \( T \) has a unique fixed point in \( X \). For each \( x \in X \), the iterative sequence \( \{T^n x\} \) is convergent to the fixed point.

**Proof.** Let \( x_0 \in X \) be fixed. Let \( x_1 = Tx_0 \), and let \( x_{n+1} = Tx_n = T^{n+1}x_0 \) for all \( n \in \mathbb{N} \).

From (2.1) with \( x = x_n \) and \( y = x_{n-1} \), we have
\[
F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \\
\leq k \{F(d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1}))\} \\
= k \{F(d(x_n, x_{n+1}) + d(x_{n-1}, x_n))\} \\
\leq k F(d(x_{n+1}, x_n)) + k F(d(x_n, x_{n-1}))
\]
which implies
\[
F(d(x_{n+1}, x_n)) \leq h F(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}, \text{ where } h = \frac{k}{1-k}.
\]

Hence
\[
F(d(x_{n+1}, x_n)) \leq h F(d(x_n, x_{n-1})) \leq h^2 F(d(x_{n-1}, x_{n-2})) \leq \cdots \leq h^n F(d(x_1, x_0)).
\]

We now show that \( \{x_n\} \) is a Cauchy sequence in \( X \).
For \( m > n \), we have that
\[
F(d(x_n, x_m)) \leq F(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)) \\
\leq F(d(x_n, x_{n+1})) + F(d(x_{n+1}, x_{n+2})) + \cdots + F(d(x_{m-1}, x_m)) \\
\leq k^n F(d(x_1, x_0)) + k^{n+1} F(d(x_1, x_0)) + \cdots + k^{m-1} F(d(x_1, x_0)) \\
\leq \frac{k^m}{1-k} F(d(x_1, x_0)) \to 0.
\]

Hence \( \lim_{n,m \to \infty} d(x_n, x_m) = 0 \) by (2). Applying Lemma 1.2, \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \).

Let \( c \in \text{int}(P) \) be given. We can choose \( N \in \mathbb{N} \) such that \( d(x_{n+1}, x_n) \ll F^{-1}(\frac{c^{1-k}}{2k}) \) and \( d(x_n, z) \ll F^{-1}(\frac{c^{1-k}}{2}) \) for all \( n > N \).
By (F2) and (F3), \( F(d(x_{n+1}, x_n)) \ll \frac{c(1-k)}{2k} \) and \( F(d(x_n, z)) \ll \frac{c(1-k)}{2} \) for all \( n > N \).

Then we have
\[
F(d(Tz, z)) \\
\leq F(d(Tz, Tx_{n-1}) + d(Tx_{n-1}, z)) \\
\leq k\{F(d(z, Tz) + d(x_{n-1}, Tx_{n-1}))\} + F(d(Tx_{n-1}, z)) \\
= k\{F(d(z, Tz) + d(x_{n-1}, x_n))\} + F(d(x_n, z)).
\]

Hence we have
\[
F(d(Tz, z)) \leq \frac{k}{1-k}F(d(x_{n-1}, x_n)) + \frac{1}{1-k}F(d(x_n, z)) \ll \frac{c}{2} + \frac{c}{2} = c.
\]

Thus, \( F(d(Tz, z)) \ll \frac{c}{n} \) for all \( n \in \mathbb{N} \), and so \( \frac{c}{n} - F(d(Tz, z)) \in P \). Since \( \frac{c}{n} \to 0 \) and \( P \) is closed, \( -F(d(Tz, z)) \in P \). Hence \( F(d(Tz, z)) = 0 \). By (F1), \( d(Tz, z) = 0 \) and so \( z = Tz \).

Assume that \( u \) is another fixed point of \( T \).

Then from (2.1) we have
\[
F(d(z, u)) \\
= F(d(Tz, Tu)) \\
\leq k\{F(d(z, Tz) + d(u, Tu))\} \\
= k\{F(d(z, z) + d(u, u))\} \\
= 0.
\]

Hence \( F(d(z, u)) \in -P \), and hence \( F(d(z, u)) = 0 \).

By (F1), \( d(z, u) = 0 \), and so \( z = u \).

Therefore, \( T \) has a unique fixed point in \( X \).

\[ \square \]

**Remark 2.1.** In Theorem 2.1, if we have \( F(t) = t \) for all \( t \in P \) then we obtain theorem 2.6 in [7].

**Theorem 2.2.** Let \( (X, d) \) be a complete cone metric space. Suppose that a mapping \( T : X \to X \) satisfies
\[
F(d(Tx, Ty)) \leq k\{F(d(y, Tx) + d(x, Ty))\}
\]
for all \( x, y \in X \), where \( k \in [0, \frac{1}{2}) \) and \( F \in \mathcal{S}(P, P) \) such that

(1) \( F \) is subadditive;

(2) if, for \( \{c_n\} \subset P \), \( \lim_{n \to \infty} F(c_n) = 0 \) then \( \lim_{n \to \infty} c_n = 0 \).

Then \( T \) has a unique fixed point in \( X \). For each \( x \in X \), the iterative sequence \( \{T^n x\} \) is convergent to the fixed point.
Proof. Let $x_0 \in X$ be fixed. Let $x_1 = Tx_0$, and let $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$.

If there exists $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then $Tx_n = x_{n+1} = x_n$, and so $T$ has a unique fixed point. Hence the proof is complete.

Thus $x_{n+1} \neq x_n$ for any $n \in \mathbb{N} \cup \{0\}$.

From (2.2) with $x = x_n$ and $y = x_{n-1}$, we have

$$F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \leq k\{F(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}))\} = k\{F(d(x_{n-1}, x_{n+1}) + d(x_n, x_n))\} \leq kF(d(x_{n-1}, x_n)) + kF(d(x_n, x_{n+1})) \leq kF(d(x_{n-1}, x_n)) + kF(d(x_n, x_{n+1})).$$

Thus we obtain

$$F(d(x_{n+1}, x_n)) \leq hF(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}, \text{ where } h = \frac{k}{1-k}.$$

Hence

$$F(d(x_{n+1}, x_n)) \leq hF(d(x_n, x_{n-1})) \leq h^2F(d(x_{n-1}, x_{n-2})) \leq \cdots \leq h^nF(d(x_1, x_0)).$$

As in proof of Theorem 2.1, $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $z \in X$ such that $\lim_{n \to \infty} x_n = z$.

Let $c \in \text{int}(P)$ be given. We can choose $N \in \mathbb{N}$ such that $d(x_n, z) \ll F^{-1}\left(\frac{c(1-k)}{3}\right)$ for all $n > N$.

By (F2) and (F3), $F(d(x_n, z)) \ll \frac{c(1-k)}{3}$ for all $n > N$.

Then we have Then we have

$$F(d(Tz, z)) \leq F(d(Tz, Tx_{n-1}) + d(Tx_{n-1}, z)) \leq k\{F(d(x_{n-1}, Tz) + d(z, Tx_{n-1}))\} + F(d(Tx_{n-1}, z)) = k\{F(d(x_{n-1}, Tz) + d(z, x_n))\} + F(d(x_n, z)) \leq k\{F(d(x_{n-1}, z) + d(z, Tz) + d(z, x_n))\} + F(d(x_n, z)).$$

Hence we have

$$F(d(Tz, z)) \leq \frac{1}{1-k}\{F(d(x_{n-1}, z) + d(z, x_n))\} + F(d(x_n, z)) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c.$$

As in proof of Theorem 2.1, we have $-F(d(Tz, z)) \in P$, and so $F(d(Tz, z)) = 0$. By F1), $d(Tz, z) = 0$. Hence $z = Tz$.

Suppose that $u$ is another fixed point of $T$ such that $u \neq z$. 
Then from (2.2) we have
\[
F(d(z, u)) = F(d(Tz, Tu)) \
\leq k\{F(d(u, Tz) + d(z, Tu))\} \
= k\{F(d(u, z) + d(z, u))\} \
\leq 2kF(d(u, z)) < F(d(u, z))
\]
which is a contradiction.

Therefore, \( T \) has a unique fixed point in \( X \).

\[ \square \]

**Remark 2.2.** In Theorem 2.2, if we have \( F(t) = t \) for all \( t \in P \) then we obtain theorem 2.7 in [7].

**Theorem 2.3.** Let \((X, d)\) be a complete cone metric space. Suppose that a mapping \( T : X \to X \) satisfies

\[
F(d(Tx, Ty)) \leq kF(d(x, y)) + lF(d(x, Ty))
\]  
(2.3)

for all \( x, y \in X \), where \( k, l \in [0, 1) \) and \( F \in \mathcal{F}(P, P) \) such that

1. \( F \) is subadditive;
2. if, for \( \{c_n\} \subset P \), \( \lim_{n \to \infty} F(c_n) = 0 \) then \( \lim_{n \to \infty} c_n = 0 \).

Then \( T \) has a fixed point in \( X \). For each \( x \in X \), the iterative sequence \( \{T^n x\} \) is convergent to the fixed point.

Moreover, if \( k + l < 1 \) then \( T \) has a unique fixed point in \( X \).

**Proof.** Let \( x_0 \in X \) be fixed. Let \( x_1 = Tx_0 \), and let \( x_{n+1} = Tx_n = T^{n+1}x_0 \) for all \( n \in \mathbb{N} \).

If there exists \( n \in \mathbb{N} \) such that \( x_{n+1} = x_n \), then \( Tx_n = x_{n+1} = x_n \), and so \( T \) has a fixed point. Hence the proof is complete.

Hence we have that \( x_{n+1} \neq x_n \) for any \( n \in \mathbb{N} \cup \{0\} \).

From (2.2) with \( x = x_n \) and \( y = x_{n-1} \), we have
\[
F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1})) \
\leq kF(d(x_n, x_{n-1})) + lF(d(x_n, Tx_{n-1})) \
= kF(d(x_n, x_{n-1})) + lF(d(x_n, x_n)) \
\leq kF(d(x_n, x_{n-1})).
\]

Thus we obtain
\[
F(d(x_{n+1}, x_n)) \leq kF(d(x_n, x_{n-1})) \text{ for all } n \in \mathbb{N}.
\]
Fixed points

Hence

\[ F(d(x_{n+1}, x_n)) \leq kF(d(x_n, x_{n-1})) \leq k^2F(d(x_{n-1}, x_{n-2})) \leq \cdots \leq k^nF(d(x_1, x_0)). \]

As in proof of Theorem 2.1, \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \).

Let \( c \in \text{int}(P) \) be given. We can choose \( N \in \mathbb{N} \) such that \( d(x_{n-1}, z) \ll F^{-1}(\xi) \) for all \( n > N \).

By (F2) and (F3), \( F(d(x_{n-1}, z)) \ll \xi \) for all \( n > N \).

Thus, for all \( n > N \), we obtain

\[
F(d(z, Tz)) \\
\leq F(d(z, x_n) + d(x_n, Tz)) \\
\leq F(d(z, x_n)) + F(d(Tx_{n-1}, Tz)) \\
\leq F(d(z, x_n)) + kF(d(x_{n-1}, z)) + lF(d(z, Tx_{n-1})) \\
= F(d(z, x_n)) + kF(d(x_{n-1}, z)) + lF(d(z, x_n)) \\
\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \\
= c.
\]

As in proof of Theorem 2.1, we have \( z = Tz \).

Suppose that \( u \) is another fixed point of \( T \).

Then from (2.3) we have

\[
F(d(z, u)) \\
= F(d(Tz, Tu)) \\
\leq kF(d(z, u)) + lF(d(z, Tu)) \\
= k\{F(d(u, z)) + lF(d(z, u))\} \\
= (k + l)F(d(u, z).
\]

Thus, \( (k + l - 1)F(d(u, z) \in P \). Since \( 0 \leq k + l < 1 \), \( (k + l - 1)F(d(u, z) \in -P \).

Hence, \( F(d(z, u)) = 0 \). By (F1), \( d(z, u) = 0 \), and so \( z = u \).

Therefore, \( T \) has a unique fixed point in \( X \). \( \square \)

Remark 2.3. In Theorem 2.3, if we have \( F(t) = t \) for all \( t \in P \) then we obtain theorem 2.8 in [7].

We now give an example to support Theorem 2.1.

Example 2.4. Let \( X = \{\frac{1}{2^n} : n = 1, 2, \ldots\} \cup \{0\} \), \( E = C^2_{\mathbb{R}}([0, 1]) \) with the norm \( \|f\| = \|f\|_{\infty} + \|f'\|_{\infty} \) and \( P = \{f \in E : f \geq 0\} \).

Then \( P \) is not normal, and hence not regular (see [7]). Define \( d : X \times X \to E \) by \( d(x, y) = |x - y|f \), where \( f : [0, 1] \to \mathbb{R} \) such that \( f(t) = e^t \).
Then \((X,d)\) is a complete cone metric space.
Let \(F(t) = t\) for all \(t \in P\) and \(k = \frac{1}{3}\).
Suppose that a mapping \(T\) is defined by
\[
Tx = \begin{cases} 
0, & x = 0, \\
\frac{1}{2n+2}, & x = \frac{1}{2^n}, n \geq 1.
\end{cases}
\]
We show that (2.1) is satisfied.
We consider the following three cases.

Case 1: \(x = y\).
Then \(F(d(Tx,Ty)) = F(0) = 0 \leq k\{F(d(x,Tx) + d(y,Ty))\}\).

Case 2: \(x = 0, y = \frac{1}{2^n}\) (or \(x = \frac{1}{2^n}, y = 0\)).
Then
\[
F(d(Tx,Ty)) = (d(0, \frac{1}{2n+2})) = (\frac{1}{2n+2}f) = \frac{1}{3} \frac{1}{2n+2}f = \frac{1}{3} \left( \frac{1}{2n} - \frac{1}{2n+2} \right) f = kF(d(y,Ty)) \leq k\{F(d(y,Ty) + d(x,Tx))\}.
\]

Case 3: \(x = \frac{1}{n}, y = \frac{1}{m}\).
Then
\[
F(d(Tx,Ty)) = (d(\frac{1}{2n+2}, \frac{1}{2m+2})) = \left| \frac{1}{2n+2} - \frac{1}{2n+2} \right| f = \frac{1}{3} \left\{ \frac{3}{2n+2} f + \frac{3}{2m+2} f \right\} = \frac{1}{3} \left\{ (\frac{1}{2n} - \frac{1}{2n+2}) f + (\frac{1}{2m} - \frac{1}{2m+2}) f \right\} = \frac{1}{3} \left\{ d(\frac{1}{2n}, \frac{1}{2n+2}) + d(\frac{1}{2m}, \frac{1}{2m+2}) \right\} = k\{F(d(x,Tx) + d(y,Ty))\}.
\]
Thus $T$ satisfies all conditions of Theorem 2.1, and so $T$ has a unique fixed point.

ACKNOWLEDGEMENTS. This research was supported by Hanseo University.

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Received: May, 2012