Note on Newton Interpolation Formula

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Abstract

We derive an interpolation formula that generalizes both Newton interpolation formula and barycentric Lagrange interpolation formula, to use interpolants that fit the derivatives, as well as the function values for arbitrary spaced grids. In addition, we introduce two variants of Newton interpolation formula. We provide numerical experiment to study the relative error analysis of the new formula and variants of Newton formula. The numerical results show that the Newton formula with barycentric weights is quite effective to interpolate functional values, when the additional information about derivatives at some points are given.

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1 Introduction

Polynomial Interpolation theory has several uses[1]. Lagrange interpolation is a well-known classical technique for interpolation. It is praised for analytic utility and beauty but deplored for numerical practice. This can be rewritten in two more computationally attractive forms: a modified Lagrange form and a barycentric form. Berrut and Trefethen (2004) have recently collected and explained the attractive features of these formulae[2]. They argue convincingly that interpolation via a barycentric Lagrange formula ought to be the standard method of polynomial interpolation. Further, Higham has given an error analysis of the evaluation of the interpolating polynomial using two formulas. His analysis shows that barycentric Lagrange interpolation should be the polynomial interpolation method of choice[5].
Newton divided difference formula [3, 4] also have several advantages. It leads to elegant methods for incorporating information on derivatives $f^{(p)}(x_j)$ when solving so-called Hermite interpolation problems. The analysis in Ref[6] and from the experiments reported in Ref[5] show that the errors from Newton form are very dependent on the ordering and can be unacceptably large even for Chebyshev points. In the present study, we introduce an interpolation formula combining Newton interpolation formula and barycentric Lagrange interpolation formula that interpolates a functional value using data known at a set of isolated points. The data may include the function values and various derivatives. Also, we derive two variants of Newton interpolation formula like variants of barycentric Lagrange interpolation formula. The numerical results show that the new formula is quite effective to interpolate functional values comparing with Newton formula and its variants, when the additional information about derivatives at some points are given.

2 Interpolation

2.1 Newton formula with barycentric weights.

Let $x_0, x_1, \ldots, x_{r-1}$ are any numbers and $x_r, x_{r+1}, x_{r+2}, \ldots, x_n$ are $n - r + 1$ distinct numbers be given, together with corresponding numbers $f_i, i = 0 : n$, which may or may not be samples of a function $f$. Define Lagrange polynomials and barycentric weights on $x_r, x_{r+1}, \ldots, x_s, 0 \leq r < s \leq n$ as follows

$$l_i^{(r,s)}(x) = \prod_{j=r,i \neq j}^s \frac{x - x_j}{x_i - x_j}, \quad l^{(r,s)}(x) = \prod_{j=r}^s x - x_j \quad \text{and} \quad w_i^{(r,s)} = \prod_{j=r,i \neq j}^s \frac{1}{x_i - x_j}$$

Let us begin by expanding $f(x)$ through Newton divided difference formula on $x_0, x_1, \ldots, x_{r-1}$

$$f(x) = \sum_{i=0}^{r-1} f[x_0, x_1, \ldots x_i]l^{(0,i-1)}(x) + f[x, x_0, \ldots, x_{r-1}]l^{(0,r-1)}(x) \quad (1)$$

Now, expanding $f[x, x_0, \ldots, x_{r-1}]$ by barycentric Lagrange interpolation formula on $x_r, x_{r+1}, x_{r+2}, \ldots, x_n$

$$f[x, x_0, x_1, \ldots, x_{r-1}] = \frac{\sum_{i=0}^{r-1} f[x_i, x_0, x_1, \ldots, x_{r-1}] \frac{w_i^{(0,r-1)}}{x - x_i}}{\sum_{i=0}^{r-1} \frac{w_i^{(0,r-1)}}{x - x_i}} + E(x) \quad (2)$$
where \( E(x) = l^{(0,r-1)}(x) \frac{f^{(n+1)}(\xi)}{(n+1)!} \). Substituting the value of \( f[x, x_0, \ldots, x_r] \) in (1) and after simplification, we find that

\[
f(x) = \sum_{i=0}^{r-1} f[x_0, x_1, \ldots, x_i] l^{(0,i-1)}(x) + \sum_{i=0}^{r-1} f[x_i, x_0, x_1, \ldots, x_{r-1}] \frac{w^{(0,r-1)}_i}{x-x_i} + E(x)
\]

where \( E(x) = l(x) \frac{f^{(n+1)}(\xi)}{(n+1)!} \). The divided differences can be determined through following new divided difference table (as shown in Table 1). The first part of the new formula in (3) needs construction of divided difference table on first \( r \) data. It requires only \( O(r^2/2) \) divisions which is independent of \( n \) whereas Newton formula needs cost of \( O(n^2/2) \) divisions. The second part of (3) requires \( O((n-r)) \) multiplications, independent of \( x \), once the numbers \( w^{(0,r-1)}_i \) are known. Hence, the cost of evaluating interpolating polynomial requires \( O(n) \) flops. Barycentric Lagrange formula too is requiring \( O(n) \) flops but does not contain divided differences whereas the new formula posses this extra feature.

| \( x_0 \) | \( f_0 \) | \( f[x_0, x_1] \) |
| \( x_1 \) | \( f_1 \) | \( f[x_0, x_1, x_2] \) |
| \( x_2 \) | \( f_2 \) | \( f[x_1, x_2, x_3] \) |
| \( x_3 \) | \( f_3 \) | \( f[x_0, x_1, x_4] \) |
| \( x_4 \) | \( f_4 \) | \( f[x_0, x_1, x_5] \) |
| \( x_5 \) | \( f_5 \) | \( f[x_0, x_5] \) |

### Table 1: Modified divided difference table.

#### 2.2 Variants of Newton interpolation.

Let \( f^{[i:i+r]} \) and \( \Gamma^{[i:i+r]} \) denote the \( r^{th} \) divided difference of the functions \( f(t) \) and \( \frac{1}{x-t} \) at the points \( x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+r} \) respectively as follow as

\[
f^{[i:i+r]} = [x_i, x_{i+1}, \ldots, x_{i+r}; f(t)] \quad \text{and} \quad \Gamma^{[i:i+r]} = [x_i, x_{i+1}, \ldots, x_{i+r}; \frac{1}{x-t}]
\]
We are interested in the problem of finding the polynomial $p_n(x)$ of degree at most $n$ that interpolates to the data $f_j$ at the points $x_j$, $j = 0 : n$. We consider fixed interpolation points $x_j$, a fixed evaluation point $x$, and a varying vector $f$. Then, using Newton’s divided difference formula at the points $x_0, x_1, \ldots, x_n$, we have

$$f(x) = p_n(x) + l^{(0,n)}(x)\frac{f^{(n+1)}(\xi)}{(n+1)!}$$

(4)

where

$$p_n(x) = \sum_{i=0}^{n} f^{[0:i]} l^{(0,i-1)}(x)$$

(5)

Since $\sum_{k=i}^{n} l^{(i,n)}(x) = 1$, (5) can be rewritten as follow as

$$p_n(x) = \sum_{i=0}^{n} f^{[0:i]} l^{(0,i-1)}(x) \times 1 = \sum_{i=0}^{n} f^{[0:i]} l^{(0,i-1)} \sum_{k=i}^{n} l^{(i,n)}(x)$$

Now, using the identity $l^{(i,n)}_k(x) = l^{(i,n)}(x) \frac{w^{(i,n)}_k}{x-x_k}$, we obtain

$$= l^{(0,n)}(x) \sum_{i=0}^{n} f^{[0:i]} \sum_{k=i}^{n} \frac{w^{(i,n)}_k}{x-x_k}$$

Since $\Gamma^{(i:n)} = \sum_{k=i}^{n} \frac{w^{(i,n)}_k}{x-x_k}$ [1], where $\Gamma^{(i,n)}$ is the $(n-i)^{th}$ divided difference of $\Gamma(t) = 1/(x-t)$ at the points $x_i, x_{i+1}, \ldots, x_n$, we have

$$p_n(x) = l^{(0,n)}(x) \sum_{i=0}^{n} f^{[0:i]} \Gamma^{[i:n]}$$

(6)

Equation (6) is the first variant of Newton divided difference formula. If we take $f(x) = 1$ in (6), then we find that

$$l^{(0,n)}(x) = \frac{1}{\Gamma^{[0:n]}}$$

Using the above identity in (6) and after simplification, we obtain

$$p_n(x) = \sum_{i=0}^{n} f^{[0:i]} \frac{\Gamma^{[i:n]}}{\Gamma^{[0:n]}}$$

(7)

Equation (7) is the second variant of Newton divided difference formula. In fact, this the true form of barycentric Newton interpolation formula. Newton variants require cost of $O(n^2)$ divisions to evaluate an interpolating polynomial. Clearly, modified Newton formula with barycentric weights in (3) has more advantages than Newton’s variants in (6) and (7).
3 Numerical results and Discussion

We give two examples to study relative error analysis of Newton interpolation formula and the new formula given in (3) for osculating interpolation and to compare the error analysis of Newton formula with its two variants. The computations were performed in MATLAB, for which $u \approx 10^{-16}$. In the first example, we have taken 25 Chebyshev first kind of points for Runge function $f(x) = \frac{1}{1+25x^2}$ on $[-1,1]$ and first five points of $f'(x) = \frac{-50x}{(1+25x^2)^2}$ (assume $r = 9$). We evaluate the interpolant at 120 equally spaced points on $[-1+10^3\epsilon, 1-10^3\epsilon]$.

![Figure 1: Relative errors in computed $f(x)$ for 25 Chebyshev first kind of points for $1/(1+25x^2)$ and its first order derivatives at first five points.](image)

where $\epsilon = 2u$ (MATLAB’s eps). The ‘exact’ values were obtained by computing 50 digit arithmetic using MATLAB’s Symbolic Math Toolbox. It is well-known that polynomial interpolation is not suitable for Runge phenomenon[7]. Here, we show the effect of small change in the functional value on the interpolant. Figure 1 plots the relative errors of Newton divided difference formula and new formula. We see that the Newton formula performs very unstable when $x$ approaches one end. But, the new formula gives better accuracy than Newton formula on the same end.

In the second example, we take 20 equally spaced points $x_j$ on $[0, 1]$ (thus $n = 19$) and $f(x) = e^x$. We evaluate the interpolant at 100 equally spaced points on $[10^3\epsilon, 1+10^3\epsilon]$. Figure 2 plots the errors for Newton divided difference form, with the latter form evaluated by nested multiplication and its two variants (6) and (7). We see that the relative errors of Newton formula and its two variants increases towards the end of the interval. We observe that the two variants of Newton interpolation formula depends on the ordering of the
data like Newton interpolation formula. Also, the numerical stability of its variants are slightly less accuracy than Newton interpolation formula. Hence, the new formula described in (3) is very handier than Newton interpolation formula and its variants.

4 Conclusion

We have derived an interpolation formula which generalizes both Newton interpolation and barycentric Lagrange interpolation formula which will include additional information about derivatives. Also, we have introduced two variants of Newton interpolation formula. Comparing with Newton and barycentric formula, the Newton-barycentric formula posses some featured advantages. The numerical results suggest that Newton-barycentric Lagrange formula is the better choice than Newton divided difference form and its two variants even in Chebyshev distributions.

References


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