Optimal Control in Linear Systems

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Abstract

In this study, a problem related with management functions for boundry and Cauchy problems in linear integro-differential systems having Fredholm Integrals is taken in hand and the solution is done with the help of using minimum Power.

Keywords: Linear Equations, Power Series, Management Systems

1. Definition and Structure of the Problem

Definition of Management Problem with Minimum Power

Process of Management

\[ L[X] \equiv x^{(n)} + \sum_{i=1}^{n} p_i(t)x^{(n-i)} = f(t)u(t) + \lambda \int_{0}^{1} \sum_{i=0}^{n} M_i(t,s)x^{(i)}(s)ds \]

Fredholm-type of linear integro- differential equation ;

\[ x(t_1) = a_1, \ x(t_2) = a_2, ..., x(t_n) = a_n \]

\[ \lambda \text{ is a parameter, } p_i(t), f(t), M_i(t,s) \text{ are known functions and } 0 \leq t \leq 1; \]

\[ 0 \leq s \leq 1; 0 \leq t_1 < t_2 < ... < t_{n-1} < t_n \leq 1, u(t) \in L^2[0,1] \text{ as acceptable management function, in the segment } [0,1] \text{ with the points } \tau_i, i = 1, ..., n, \]

\[ 0 \leq \tau_1 < \tau_2 < ... < \tau_{n-1} < \tau_n = 1, \text{ and current conditions for this points} \]

\[ x(\tau_1) = b_1, x(\tau_2) = b_2, ..., x(\tau_n) = b_n \]

is put forward with the above definition. Previously, although the manageable object is stated with (1)-(3) conditions, the minimum energy usage problem is solved in the study no [1]. In the study functional equations written as Moment equalities form and taken in hand by N.N.Krasovskiy [2] are used. K.A.Kasimov’s [3] method was applied for the solution of the values of Linear integro-differential equation. Here, one of the problems that we want to solve
is as below: for acceptable \( u(t) \in L^2[0,1] \) let’s find a function \( u = u^0(t) \) among the management functions so that the function (corresponding it) \( x = x(t) \), which is the solution of the boundary problems (1) and (2), provides the condition no (3) and:

\[
J[u] = \|u\|_{C[0,1]}^2 = \max_{0 \leq t \leq 1} |u(t)| = \min
\]

i.e. let the functional \( J[u] \) has the smallest value.

2. Solution

2.1. Solution of the Boundary Problem

According to the study [3], and for \( p_i(t) \in C[0,1] \), n-th degree linear differential equation

\[
L[x] = 0 \tag{5}
\]

in the interval \([0,1]\) there exists linear independent \( x_1(t), x_2(t), ..., x_n(t) \) solutions (i.e. base solutions). Corresponding this, if \( W[s] \) Wronski determinant is expressed in the following format:

\[
W[s] = \det \begin{pmatrix} x_1(s), x_2(s), ..., x_n(s) \\ x_1'(s), x_2'(s), ..., x_n'(s) \\ \vdots \\ x_1^{(n-1)}(s), x_2^{(n-2)}(s), ..., x_n^{(n-1)}(s) \end{pmatrix} \tag{6}
\]

and if we put \( x_1(t), x_2(t), ..., x_n(t) \) base solutions of the Eq. (5) instead of \( i \)-th line \( x_1^{(i-1)}(s), x_2^{(i-1)}(s), ..., x_n^{(i-1)}(s) \) of \( W[s] \) determinant, we obtain a new determinant. If we show this determinant as \( W_i[t,s] \)

\[
W_i[t,s] = \det \begin{pmatrix} x_1(s), x_2(s), ..., x_n(s) \\ \vdots \\ x_1^{(i-2)}(s), x_2^{(i-2)}(s), ..., x_n^{(i-2)}(s) \\ x_1(t), x_2(t), ..., x_n(t) \\ x_1^{(i)}(s), x_2^{(i)}(s), ..., x_n^{(i)}(s) \\ \vdots \\ x_1^{(n-1)}(s), x_2^{(n-2)}(s), ..., x_n^{(n-1)}(s) \end{pmatrix} \tag{7}
\]

\( i = 1, 2, ..., n \) is written.

On the other hand if we consider the Cauchy function:

\[
K_i(t,s) = W^{-1}[s]W_i[t,s], \ i = 1, 2, ..., n. \tag{8}
\]

is written and so Cauchy functions that is defined with the help of the Eq. (6)-(8) satisfies the Eq. (9):

\[
K_i^{(j)}(s,s) = \begin{cases} 1, & j = i - 1 \\ 0, & j \neq i - 1, j = 0,1,...,n-1 \end{cases} \tag{9}
\]
Let’s Analyze the Specific Functions of Boundary Problem [1]:
The functions of specific boundary problem given in the Eq. (1),(2):
\[ \varphi_1(t) = \Delta^{-1} \cdot \Delta_1(t), \quad i = 1, 2, ..., n \] (10)
Here
\[ \Delta = \det \begin{pmatrix} x_1(t_1), x_2(t_1), ..., x_n(t_1) \\ x_1(t_2), x_2(t_2), ..., x_n(t_2) \\ \vdots \\ x_1(t_n), x_2(t_n), ..., x_n(t_n) \end{pmatrix} \] (11)
is the determinant that occurred after putting the line consisting of base solutions \( x_1(t), x_2(t), ..., x_n(t) \) of the Eq. (5) instead of \( i \)-th line \( x_1(t_1), x_2(t_1), ..., x_n(t_1) \) of \( \Delta \) determinant where \( \Delta(t), i = 1, 2, ..., n \). If we return the Eq. (10): \( \delta \) - as the symbol of Kronecker
\[ \varphi_1(t_j) = \delta_{ij} = \begin{cases} 1, & i = j \ i.e. \\ 0, & i \neq j \ i.e. \end{cases} \] (12)
Solution of the Problem (2): For the solution:
\[ x(t) = \sum_{k=1}^{n} c_k \varphi(t) + \int_0^1 K_n(t, s) \Phi(s) ds \] (13)
is written. Here \( \varphi_k(t), k = 1, 2, ..., n \), are specific functions of the boundary problems (1),(2) which is defined with the Eq. (10), \( K_n(t, s) \) is Cauchy’s n-th function, \( c_1, c_2, ..., c_n \) - are unknown constants, the right-side of \( \Phi(t) \) with (1) integro-differential equations are defined and :
\[ \Phi(t) = f(t)u(t) + \lambda \int_0^1 \sum_{i=0}^{n} M_i(t, s) x^{(i)}(s) ds \] (14)
is written. If we put the function \( x(t) \) from the Eq. (13) to its place in the Eq. (14), then according to the unknown function \( \Phi(t) \), we find the following Fredholm Integral equation
\[ \Phi(t) = f(t)u(t) + \lambda \sum_{k=1}^{n} c_k \alpha_k(t) + \lambda \int_0^1 H(t, s) \Phi(s) ds. \] (15)
Here
\[ \alpha_k(t) = \int_0^1 \sum_{i=0}^{n} M_i(t, \tau) \varphi_k^{(i)}(\tau) d\tau, \quad k = 1, 2, ..., n, \] (16)
and,
\[ H(t, s) = \int_0^1 \sum_{i=0}^{n} M_i(t, \tau) K_n^{(i)}(\tau, s) d\tau \] (17)
is the kernel of the Fredholm Integral equation that is given in the Eq. (15).

Now, if we accept that the number $\lambda$ is not the eigenvalue of the kernel $H(t,s)$ defined with the Eq. (17) given above and $R(t,s,\lambda)$ function is the Resolvent of this kernel, then the solution of the Fredholm Integral equation given in the Eq. (14) can be written as below:

$$\Phi(t) = f(t)u(t) + \lambda \sum_{k=1}^{n} c_k \alpha_k(t) + \lambda \int_{0}^{1} R(t,s,\lambda) \left[ f(s)u(s) + \lambda \sum_{k=1}^{n} c_k \alpha_k(s) \right] ds$$

and so we get the equation

$$\Phi(t) = f(t)u(t) + \lambda \sum_{k=1}^{n} c_k \beta_k(t) + \lambda \int_{0}^{1} R(t,s,\lambda)f(s)u(s)ds \tag{18}$$

and

$$\beta_k(t) = \alpha_k(t) + \lambda \int_{0}^{1} R(t,s,\lambda)\alpha_k(s)ds, \; k = 1, 2, ..., n \tag{19}$$

the functions $\alpha_k(t), \; k = 1, 2, ..., n,$ are shown in the Eq. (16).

If we put $\Phi(t)$ function to its place in the Eq. (13) by using the Eq. (18), then the solution of the boundary problems (1), (2), i.e.

$$x(t) = \sum_{k=1}^{n} c_k \gamma_k(t) + \int_{0}^{1} Q_n(t,\tau,\lambda)f(\tau)u(\tau)d\tau \tag{20}$$

is found. In this expression:

$$\gamma_k(t) = \varphi_k(t) + \lambda \int_{0}^{1} K_n(t,s)\beta_k(s)ds \tag{21}$$

and

$$Q_n(t,\tau,\lambda) = K_n(t,\tau) + \lambda \int_{0}^{1} K_n(t,s)R(s,\tau,\lambda)ds \tag{22}$$

If we put $x(t)$ function stated with the Eq. (20) - (22) given above to its place in the boundary conditions and take into account of the Eq. (12), then we get the linear equation

$$\sum_{k=1}^{n} (\delta_{ik} + \lambda \Delta_{ik})c_k = a_i - Q_{ni}(\lambda), \; i = 1, 2, ..., n. \tag{23}$$

and here $\delta_{ij}$ - as the symbol of Kronecker, $\Delta_{ik}$:

$$\Delta_{ik} = \int_{0}^{1} K_n(t_i,s)\beta_k(s)ds, \; i, k = 1, 2, ..., n,$$
and the functions $\beta_k(t)$, $k = 1, 2, ..., n$, defined by the Eq. (19) are stated with the following equations

$$Q_n(\lambda) = \int_0^1 Q_n(t_i, s, \lambda) f(s) u(s) ds$$

The function $Q_n(t, s, \lambda)$ written there is explained by the Eq. (22).

On the other hand the determinant of the matrix $\omega = \{(\delta_{ik} + \lambda \Delta_{ik})\}_{i,k}^n$ is different from 0 $\det \omega \neq 0$ i.e. if we put forward that the inverse matrix $\omega^{-1}$ exists; if we show the term $\omega_{kj}$ in the intersection of k-th line and j-th column, then we find the solution of the linear equation system (23) as follows

$$c_k = \sum_{j=1}^n (a_j - Q_{nj}(\lambda)) \omega_{kj}, k = 1, 2, ..., n.$$  

If we put the values of $c_k$ into its place in the Eq. (20), then we convert the solution of the boundary problem (1),(2) to the following

$$x(t) = \sum_{j,k=1}^n a_j \omega_{kj} \gamma_k(t) + \int_0^1 Q_n(t, \tau, \lambda) f(\tau) u(\tau) d\tau -$$

$$- \sum_{j,k=1}^n \omega_{kj} \gamma_k(t) \int_0^1 Q_n(t_j, \tau, \lambda) f(\tau) u(\tau) d\tau.$$  

(24)

### 2.2. Arrangement of Optimal Management.

The solution of the boundary problem (1),(2)was proved by the Eq. (24) and the functions $\gamma_k(t), Q_n(t, s, \lambda)$ were defined with the help of the Eq. (21),(22).

Now, let’s put forward there exists a management function $u(t) \in L^2[0,1]$ satisfying boundary conditions in the Eq. (4) and (3). In this case if we put $Bu$ durumda $x(t)$ function from the Eq. (24) into the Eq. (3) we find the following

$$b_i - \sum_{j,k=1}^n \gamma_k(\tau_i) a_j \omega_{kj} =$$

$$= \int_0^1 \left\{ Q_n(t_i, s, \lambda) - \sum_{j,k=1}^n \gamma_k(t_i) \omega_{kj} Q_n(t_j, s, \lambda) \right\} f(s) u(s) ds$$  

(25)

To provide ease of the solution:

$$A_i = b_i - \sum_{j,k=1}^n \gamma_k(\tau_i) a_j \omega_{kj}, \ i = 1, 2, ..., n,$$

$$h_i(s, \lambda) = \left\{ Q_n(t, s, \lambda) - \sum_{j,k=1}^n \gamma_k(t_i) \omega_{kj} Q_n(t_j, s, \lambda) \right\} f(s), \ i = 1, 2, ..., n$$  

(26)
let’s state by defining the symbols. In this case we can write the equation system (25) as below

\[ \int_0^1 h_i(s, \lambda)u(s)ds = A_i, \quad i = 1, 2, ..., n \] (27)

or shortly

\[ (h_i, u) = A_i, \quad i = 1, 2, ..., n \] (28)

as Moment equations format. The symbol (\(\ast, \ast\)) is Scalar product defined as left side of the equation of (27) \(L^2[0, 1]\) Hilbert space. \(\xi_1, \xi_2, ..., \xi_n\) - arbitrary numbers, functions \(h_1, h_2, ..., h_n\) defined by (26), then

\[ h = \sum_{i=1}^{n} \xi_i h_i \] (29)

and let’s show the Linear Combinations’ s sentence in the space \(L^2[0, 1]\) as \(H^0\). Here easily the space \(H^0\) can be seen as the subspace of \(L^2[0, 1][2, 4]\). In this case for all \(u \in L^2[0, 1]\) it is possible to write as a union form as below

\[ u = h + g, \quad h \in H, g \perp H \] (30)

Since \(g \perp H\), from (28) and (30) we get

\[ (h_i, u) = (h_i, h) + (h_i, g) = (h_i, h), \quad i = 1, 2, ..., n \]

So \((h_i, g), \quad i = 1, 2, ..., n\) and the component \(g\) of element \(u\) has no effect for the element \(u\) to satisfy (28). For this reason; with minimum Power usage solution of the management problem 
\(u = u^0(t)\), if exists, belongs to \(H^0\) space and the Eq. (29) can be shown by the formula

\[ u^0 = \sum_{i=1}^{n} \xi_i h_i \] (31)

Since the element (31) of \(H^0\) space satisfied the conditions in (28), the vector \(\xi = (\xi_1, \xi_2, ..., \xi_n)\)

\[ M\xi = D \] (32)

become the solution of matrix equation. Here \(M = \{(h_i, h_j)\}_{i,j=1}^{n}\) known positive matrix, i.e. \(L^2_+\) is Hilbert space, for all \(\xi = (\xi_1, \xi_2, ..., \xi_n) \in L^2_+\), \((M\xi, \xi) > 0\) and \(D = (A_1, A_2, ..., A_n)^T\), the numbers \(A_i\) were given with the help of the equations in (26). Let’s write the equation

\[ \sum_{i=1}^{n} \lambda_i A_i = 1 \] (33)
and analyze the real constants $\lambda_i$, $i = 1, 2, ..., n$ that forms the above equation. It can be expressed easily that:

$$
\sum_{i=1}^{n} \lambda_i A_i = \int_0^1 h(t, \lambda)u(t)dt.
$$

and with the help of (33) we can write the inequality below:

$$
1 \leq \max_{0 \leq i \leq 1} |u(t)| \int_0^1 |h(t, \lambda)|dt
$$

But this inequality valid whenever every $\lambda = \lambda_i \in L^+_2$, $(i = 1, 2, ..., n)$ satisfies (33).

If we state the formula

$$
u^0(t) = A \text{sign} h(t, \lambda)
$$

then, it is satisfied for management function $u(t)$ that is necessary and sufficient condition to use the symbol (=) in the inequality (34): In the Eq. (35) $\text{sign} h(t, \lambda)$ function shows the sign of $h(t, \lambda)$ and:

$$
\text{sign} h(t, \lambda) = \begin{cases} +1, & h(t, \lambda) > 0, \\ -1, & h(t, \lambda) < 0. \end{cases}
$$

is written so that the theorem given below is proved:

**Theorem 1:**

1) $p_i(t), f(t), M(t, s) \in C(D), (D = 0 \leq t \leq 1; 0 \leq s \leq 1)$;
2) $\lambda$ - parameter is not the eigenvalue of the function $H(t, s)$ which is the kernel of Fredholm integral equation given in (38);
3) The determinant $\Delta$ stated with (11) and the matrix $\omega$ which is formed from unknown coefficients of the Eq. (23) are different from zero.

In this case for all given $(a_1, a_2, ..., a_n)$ and $b_1, b_2, ..., b_n$ vectors ,for the satisfaction of the conditions (2), (3), (4) and the solution of the linear integro-differential Eq. (1), i.e. for the solution of management problem by using minimum power to be possible and unique the necessary and sufficient condition is that the functions $h_i(t, \lambda), i = 1, 2, ..., n$ in the interval $[0, 1]$ should be linear independent. If the vector consisting of the numbers $A_i$ in the Eq. (26) is $D = (A_1, A_2, ..., A_n) \in L^+_2$, then there exists optimal management function $u^0(t)$ and it is stated by the Eq. (34). $h(t, \lambda)$ from (29) , its coefficients $\lambda_i, i = 1, 2, ..., n$ from (33) and the coefficient $A$ in the Eq. (34) can be stated with the following equality:

$$A = \left\{ \int_0^1 |h(t, \lambda)|dt \right\}^{-1}.$$
2.3 Minimum Power-Use Manager with Cauchy Problem Content

Let’s take in hand Cauchy problem for the linear-integro differential Eq. (1).

If we put forward the given conditions below:

\[ x(0) = a_1, x'(0) = a_2, \ldots, x^{(n-1)}(0) = a_n \]  

in this case the solution of Cauchy problems (1),(36) is in the form of the following

\[ x(t) = \sum_{j=1}^{n} a_j K_j(t, 0) + \int_{0}^{1} K_n(t, s) \Phi(s) ds \]  

Here \( \Phi(t) \) is defined by (14) and \( K_j(t, s), j = 1, 2, \ldots, n \) are the Cauchy functions for (5) and they were defined in (8).

If we put the function given in (37) into its place in (14), then for the function \( \Phi(t) \) the integral equation:

\[ \Phi(t) = f(t) u(t) + \lambda \sum_{j=1}^{n} a_j \int_{0}^{1} \sum_{i=0}^{n} M_i(t, s) K_j^{(i)}(s, 0) ds + \]

\[ + \lambda \int_{0}^{1} \int_{0}^{s} \sum_{i=0}^{n} M_i(t, s) K_n^{(i)}(s, \tau) \Phi(\tau) d\tau ds \]  

can be written and

\[ \tilde{\alpha}_j(t) = \int_{0}^{1} \sum_{i=0}^{n} M_i(t, s) K_j^{(i)}(s, 0) ds, \quad j = 1, 2, \ldots, n, \]

\[ \tilde{H}(t, \tau) = \int_{\tau}^{1} \sum_{i=0}^{n} M_i(t, s) K_n^{(i)}(s, \tau) ds \]  

In this case we can transform the integral equation given in (38) to Fredholm integral equation form given below:

\[ \Phi(t) = f(t) u(t) + \lambda \sum_{j=1}^{n} a_j \tilde{\alpha}_j(t) + \lambda \int_{0}^{1} \tilde{H}(t, \tau) \Phi(\tau) d\tau. \]  

In the given equation, \( \Phi(t) \) is an unknown function. Let’s put forward that \( \lambda \) is not the eigenvalue of the kernel \( \tilde{H}(t, \tau) \) stated in (37) and the function \( \tilde{R}(t, \tau, \lambda) \) is Resolven corresponding to this kernel. In this case the solution of the equation in (40) is like

\[ \Phi(t) = \lambda \sum_{j=1}^{n} a_j \left[ \tilde{\alpha}_j(t) + \lambda \int_{0}^{1} \tilde{R}(t, \tau, \lambda) \tilde{\alpha}_j(\tau) d\tau \right] + \]
\[ + f(t)u(t) + \lambda \int_0^1 \tilde{R}(t, \tau, \lambda) f(\tau)u(\tau) d\tau \]

or

\[ \Phi(t) = \lambda \sum_{j=1}^n a_j \tilde{\beta}_j(t) + f(t)u(t) + \lambda \int_0^1 \tilde{R}(t, \tau, \lambda) f(\tau)u(\tau) d\tau \]  \hspace{1cm} (41)

Here:

\[ \tilde{\beta}_k(t) = \tilde{\alpha}_k(t) + \lambda \int_0^1 \tilde{R}(t, s, \lambda) \tilde{\alpha}_k ds \]

is defined like that.

If we put \( \Phi(t) \) function stated in (41) to its place in (37), then we find

\[ x(t) = \sum_{j=1}^n a_j K_j(t, 0) + \int_0^1 K_n(t, s) \left\{ \lambda \sum_{j=1}^n a_j \tilde{\beta}_j(s) + \\
+ f(s)u(s) + \lambda \int_0^1 \tilde{R}(s, \tau, \lambda) f(\tau)u(\tau) d\tau ds \right\} + \int_0^1 K_n(t, s) f(s)u(s) ds + \]

\[ + \lambda \int_0^1 K_n(t, s) \int_0^1 \tilde{R}(s, \tau, \lambda) f(\tau)u(\tau) d\tau ds \]  \hspace{1cm} (42)

if we define the following

\[ K_J(t, 0) + \lambda \int_0^1 K_n(t, s) \tilde{\beta}_j(s) ds = \tilde{\gamma}_j(t) \]

then the equality (42) is transformed to the form

\[ x(t) = \sum_{j=1}^n a_j \tilde{\gamma}_j(t) + \int_0^1 K_n(t, s) f(s)u(s) ds + \\
+ \lambda \int_0^1 K_n(t, s) \int_0^1 \tilde{R}(s, \tau, \lambda) f(\tau)u(\tau) d\tau ds \]  \hspace{1cm} (43)

### 2.4. Minimum Power-Use Manager For Cauchy Problem

Suppose that the following conditions

\[ x(1) = b_1, \quad x'(1) = b_2, \ldots, x^{(n-1)}(1) = b_n \]  \hspace{1cm} (44)
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were shown at the right end of the interval $[0, 1]$ i.e. at the point $t = 1$. We can state this minimum power-use problem as below:

We need to find such a $u = u^0(t)$ manager function among the acceptable manager functions $u(t) \in L^2[0, 1]$ that (corresponding it) the function $x = x(t)$ which is both given by the formula (43) and the solution of the Cauchy problems (1),(34), satisfies the conditions and the functional (4) has the smallest value.

If we both take $n - 1$-th derivative of both sides of the Eq. (43) and take into account the properties of $K_n(t, s)$ (i.e. Cauchy functions) given by Eq. (9), then:

$$x^{(i)}(t) = \sum_{j=1}^{n} a_j \tilde{\gamma}^{(i)}(t) + \int_{0}^{1} K_n^{(i)}(t, s) f(s) u(s) ds +$$

$$+ \lambda \int_{0}^{1} K_n^{(i)}(t, s) \int_{0}^{1} \tilde{R}(s, \tau, \lambda) f(\tau) u(\tau) d\tau ds, \ i = 1, 2, ..., n - 1,$$

is written. This expression from Eq. (43),(44)

$$\tilde{\gamma}_i = -\sum_{j=1}^{n} a_j \tilde{\gamma}^{(i-1)}(1) = \int_{0}^{1} K_n^{(i-1)}(1, s) f(s) u(s) ds +$$

$$+ \lambda \int_{0}^{1} K_n^{(i-1)}(1, s) \int_{0}^{1} \tilde{R}(s, \tau, \lambda) f(\tau) u(\tau) d\tau ds, \ i = 1, 2, ..., n$$

or

$$\tilde{A}_i = \int_{0}^{1} [K_n^{(i-1)}(1, s) + \lambda \int_{0}^{1} K_n^{(i-1)}(1, \tau) \tilde{R}(\tau, s, \lambda) d\tau] f(s) u(s) ds \quad (45)$$

is written. Here

$$\tilde{A}_i = \tilde{\gamma}_i - \sum_{j=1}^{n} a_j \tilde{\gamma}^{(i-1)}(1), \ i = 1, 2, ..., n$$

Now if we take into account of the below expression

$$\tilde{h}_i(s, \lambda) = [K_n^{(i-1)}(1, s) + \lambda \int_{0}^{1} K_n^{(i-1)}(1, \tau) \tilde{R}(\tau, s, \lambda) d\tau] f(s), \ i = 1, 2, ..., n$$

then we can write the Eq. (45) as integral equality:

$$\int_{0}^{1} \tilde{h}_i(s, \lambda) u(s) ds = \tilde{A}_i, \ i = 1, 2, ..., n \quad (46)$$

We can write the conditions given by (44)

$$(\tilde{h}_i, u) = \tilde{A}_i, \ i = 1, 2, ..., n \quad (47)$$
as Moment equalities \([2, 4]\). Here:

\[
(\tilde{h}_i, u) = \int_0^1 \tilde{h}_i(s, \lambda) u(s) ds,
\]

is the Scalar Product in \(L^2[0, 1]\) Hilbert space.

We can also find manager by using minimum power for Cauchy problems \((1),(34)\) expressions as equalant to ideas given in arrangements of the optimal manager. So that the following theorem is proved.

**Theorem 2.**

1) \(p_i(t), f(t), M_i(t, s) \in C(D), (D = 0 \leq t \leq 1; 0 \leq s \leq 1)\);

2) Let's put forward that \(\lambda\) - parameter is not the eigenvalue of the function \(\tilde{H}(t, s)\) which is the kernel of Fredholm integral equation given in Eq. \((40)\). In this case for all given \((a_1, a_2, ..., a_n)\) and \((b_1, b_2, ..., b_n)\) vectors for the solution of the integro- differential equation \((1)\) satisfying the conditions of \((34),(42),(4)\) (i.e. for the solution of management problem) to be valid and unique, the necessary and sufficient condition is that the functions \(\tilde{h}_i(t, \lambda), i = 1, 2, ..., n\) in the interval \([0, 1]\) should be linear independent.

In this case the solution of Moment equation system in Eq. \((47)\) is as below:

\[
\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, ..., \tilde{\xi}_n) \text{ vector is the solution of the matrix equation}
\]

\[
\tilde{M}\tilde{\xi} = \tilde{A}
\]

and the matrix

\[
M = \left\{(\tilde{h}_i, \tilde{h}_j)\right\}_{i,j=1}^n
\]

is known matrix.

**3. Conclusion:** At the end of this study, the structure of the problem has been stated. The optimal manager with its boundary values has been arranged and Minimum Power-Use Manager with Cauchy Content and its solution has been satisfied. So the Hypothesis has been confirmed.

**References**


Received: August, 2011