Intuitionistic Q-Fuzzy Bi-ideals of Ordered Semigroups

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Abstract

The intuitionistic Q-fuzzification of the notion of a bi-ideal in ordered semigroups is considered. In terms of intuitionistic Q-fuzzy set, conditions for an ordered semigroup to be completely regular is provided. Characterizations of intuitionistic Q-fuzzy bi-ideals in ordered semigroups are given.

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1 Introduction

After the introduction of fuzzy sets by L. A. Zadeh [8], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by K.T. Atanssov [1,2], as a generalization of the notion of fuzzy set. In [3], N. Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. In this paper the concept of Q-fuzzy bi-ideals of an ordered semigroup has been introduced. We consider the Q-fuzzification of the notion of a bi-ideal in ordered semigroups. We show that every intuitionistic Q-fuzzy bi-ideal of an ordered semigroup is an intuitionistic Q-fuzzy subsemigroup. We prove that, in a regular, left and right simple ordered semigroup, every intuitionistic Q-fuzzy bi-ideal is constant.

2 Preliminary Notes

In this section we discuss some elementary definitions that we use in the sequel.

An ordered semigroup we mean an ordered set $S$ at the same time a semigroup satisfying the following conditions:

$$(\forall a, b, x \in S)(a \leq b) \Rightarrow xa \leq xb \text{ and } ax \leq bx$$
A nonempty subset $A$ of an ordered semigroup $S$ is called a left (resp. right) ideal of $S$ if it satisfies:

1. $SA \subseteq A$ (resp. $AS \subseteq A$),
2. $(\forall a \in A)(\forall b \in S)(b \leq a \Rightarrow b \in A)$.

Both a left and right ideal of $S$ is called an ideal of $S$. A nonempty subset $A$ of an ordered semigroup $S$ is called a bi-ideal of $S$ if it satisfies:

1. $ASA \subseteq A$;
2. $(\forall a \in A)(\forall b \in S)(b \leq a \Rightarrow b \in A)$.

An ordered semigroup $S$ is said to be left (resp. right) simple if for every left (resp. right) ideal $A$ of $S$, we have $A = S$. An ordered semigroup $S$ is said to be regular if for every $a \in S$ there exists $x \in S$ such that $a = axa$.

A mapping $: S \times Q \rightarrow [0; 1]$, where $S$ and $Q$ are arbitrary non-empty sets, is called a $Q$-fuzzy set in $S$. A $Q$-fuzzy set $\mu$ in an ordered semigroup $S$ is called a $Q$-fuzzy subsemigroup of $S$ if $\mu(xy, q) \geq \min\{\mu(x, q), \mu(y, q)\}$ for all $x, y \in S$ and for all $q \in Q$. A $Q$-fuzzy set $\mu$ in an ordered semigroup $S$ is called a $Q$-fuzzy bi-ideal of $S$ if it satisfies:

1. $(\forall x, y \in S, \forall q \in Q)(x \leq y) \Rightarrow \mu(x, q) \geq \mu(y, q))$;
2. $(\forall x, y, z \in S, \forall q \in Q)(\mu(xyz, q) \geq \min\{\mu(x, q); \mu(z, q)\})$.

As an important generalization of the notion of $Q$-fuzzy sets in $S$. An intuitionistic $Q$-fuzzy set (IQFS for short) defined on a non-empty set $S$ as objects having the form

$$A = \{< (x, q); \mu_A(x, q); \gamma_A(x, q) > | x \in S, q \in Q\};$$

where the functions $\mu_A : S \times Q \rightarrow [0; 1]$ and $\gamma_A : S \times Q \rightarrow [0; 1]$ denote the degree of membership (namely $\mu_A(x, q)$) and the degree of nonmembership (namely $\gamma_A(x, q)$) of each element $x \in S$ and $q \in Q$ to the set $A$ respectively, and $0 \leq \mu_A(x, q) + \gamma_A(x, q) \leq 1$ for all $x \in S, q \in Q$.

### 3 Main Results

For the sake of simplicity, we shall use the symbol $A =< \mu_A; \gamma_A >$ for the intuitionistic $Q$-fuzzy set $A = \{< (x, q); \mu_A(x, q); \gamma_A(x, q) > | x \in S, q \in Q\}$.

We first consider the intuitionistic $Q$-fuzzification of the notion of bi-ideals in an ordered semigroup as follows.

**Definition 3.1** An IQFS $A =< \mu_A; \gamma_A >$ in an ordered semigroup $S$ is called an intuitionistic $Q$-fuzzy subsemigroup of $S$ if it satisfies:

1. $(\forall x, y \in S, \forall q \in Q)(\mu_A(xy, q) \geq \min\{\mu_A(x, q); \mu_A(z, q)\})$;
2. $(\forall x, y \in S, \forall q \in Q)(\gamma_A(xy, q) \leq \max\{\gamma_A(x, q); \gamma_A(z, q)\})$. 
**Definition 3.2** An IQFS $A = < \mu_A; \gamma_A >$ in an ordered semigroup $S$ is called an intuitionistic $Q$-fuzzy bi-ideal of $S$ if it satisfies:

(i) $(\forall x, y, z \in S, \forall q \in Q)(\mu_A(xyz, q) \geq \min\{\mu_A(x, q); \mu_A(z, q)\});$

(ii) $(\forall x, y, z \in S, \forall q \in Q)(\gamma_A(xyz, q) \leq \max\{\gamma_A(x, q); \gamma_A(z, q)\});$

(iii) $(\forall x, y \in S)(x \leq y) \Rightarrow \mu_A(x, q) \geq \mu_A(y, q); \gamma_A(x, q) \leq \gamma_A(y, q)).$

**Theorem 3.3** Every intuitionistic $Q$-fuzzy bi-ideal of a regular ordered semigroup $(S; \cdot, \leq)$ is an intuitionistic $Q$-fuzzy subsemigroup of $S$.

**Proof.** Let $A = < \mu_A; \gamma_A >$ be an intuitionistic $Q$-fuzzy bi-ideal of a regular ordered semigroup $S$ and let $a$ and $b$ be any elements of $S$ and $q$ be any element in $Q$. Since $S$ is regular, there exists $x \in S$ such that $b \leq bxb$. Then we have

$$\mu_A(ab, q) \geq \mu_A(a(bxb), q) = \mu_A(a(bx)b, q) \geq \min\{\mu_A(a, q); \mu_A(b, q)\}$$

and

$$\gamma_A(ab, q) \leq \gamma_A(a(bxb), q) = \gamma_A(a(bx)b, q) \leq \max\{\gamma_A(a, q); \gamma_A(b, q)\}.$$

This means that $A = < \mu_A; \gamma_A >$ is an intuitionistic $Q$-fuzzy subsemigroup of $S$. ■

**Proposition 3.4** Let $S$ be a regular ordered semigroup and let $A = < \mu_A; \gamma_A >$ be an intuitionistic $Q$-fuzzy bi-ideal of $S$: Then we have

$$(\forall a \in S, \forall q \in Q)(a \leq a^2 \Rightarrow \mu_A(a, q) = \mu_A(a^2, q); \gamma_A(a, q) = \gamma_A(a^2, q)).$$

**Proof.** Let $a \in S, q \in Q$. Then, since $S$ is regular, there exists $x \in S$ such that $a \leq axa$. Then

$$\mu_A(a, q) \geq \mu_A(a^2) \geq \mu_A(a(ax)a, q) \geq \min\{\mu_A(a, q); \mu_A(a, q)\} = \mu_A(a, q)$$

and

$$\gamma_A(a, q) \leq \gamma_A(a^2) \leq \gamma_A(a(ax)a, q) \leq \max\{\gamma_A(a, q); \gamma_A(a, q)\} = \gamma_A(a, q).$$

Hence $\mu_A(a, q) = \mu_A(a^2, q)$ and $\gamma_A(a, q) = \gamma_A(a^2, q)$. ■

**Proposition 3.5** Let $S$ be an ordered semigroup such that

(i) $(\forall x \in S)(x \leq x^2);$  
(ii) $(\forall a, b \in S)(ab \in (baS] \cap (Sba)).$

Then every intuitionistic $Q$-fuzzy bi-ideal $A = < \mu_A; \gamma_A >$ of $S$ satisfies the following condition

$$(\forall a, b \in S, \forall q \in Q)(\mu_A(ab, q) = \mu_A(ba, q); \gamma_A(ab, q) = \gamma_A(ba, q)).$$
Proof. Since \( ab \in (baS) \cap (Sba) \), we have \( ab \in (baS) \) and so \( ab \leq bax \) for some \( x \in S \). Using (ii), we get \( (ba)x \in (xbaS) \cap (Sxba) \) and thus \( bax \leq yxba \) for some \( y \in S \). It follows from (i) that

\[
ab \leq (ba)x \leq (ba)^2x = ba(bax) \leq ba(yxba)
\]

so that

\[
\mu_A(ab, q) \geq \mu_A(ba(yxba), q) = \mu_A((ba)(yx)(ba), q)
\]

\[
\geq \min\{\mu_A(ba, q), \mu_A(ba, q)\} = \mu_A(ba, q)
\]

and

\[
\gamma_A(ab, q) \leq \gamma_A(ba(yxba), q) = \gamma_A((ba)(yx)(ba), q)
\]

\[
\leq \max\{\gamma_A(ba, q), \gamma_A(ba, q)\} = \gamma_A(ba, q).
\]

The reverse inequalities are by symmetry. This completes the proof. \( \blacksquare \)

An ordered semigroup \( S \) is said to be left (resp. right) regular \([5]\) if for every \( a \in S \) there exists \( x \in S \) such that \( a \leq xa^2 \) (resp. \( a \leq a^2x \)). An ordered semigroup \( S \) is said to be completely regular \([4]\) if it is regular, left regular and right regular.

Lemma 3.6 \([6]\) An ordered semigroup \( S \) is completely regular if and only if for every \( A \subseteq S \), we have \( A \subseteq (A^2SA^2) \).

Proposition 3.7 Let \( S \) be a completely regular ordered semigroup. For every intuitionistic \( Q \)-fuzzy bi-ideal \( A = \langle \mu_A; \gamma_A \rangle \) of \( S \), we have

\[
(\forall x \in S, \forall q \in Q)(\mu_A(x, q) = \mu_A(x^2, q); \gamma_A(x, q) = \gamma_A(x^2, q))
\]

Proof. Let \( A = \langle \mu_A; \gamma_A \rangle \) be an intuitionistic \( Q \)-fuzzy bi-ideal of \( S \) and let \( x \in S, q \in Q \). Since \( x \in (x^2Sx^2) \) by Lemma 3.6, there exists \( a \in S \) such that \( x \leq x^2ax^2 \). Since \( A = \langle \mu_A; \gamma_A \rangle \) is an intuitionistic \( Q \)-fuzzy bi-ideal of \( S \), it follows that

\[
\mu_A(x, q) \geq \mu_A(x^2ax^2) \geq \min\{\mu_A(x^2, q), \mu_A(x^2, q)\}
\]

\[
= \mu_A(x^2, q) = \mu_A(xx, q) \geq \min\{\mu_A(x, q), \mu_A(x, q)\} = \mu_A(x, q)
\]

and

\[
\gamma_A(x, q) \leq \gamma_A(x^2ax^2) \leq \max\{\gamma_A(x^2, q), \gamma_A(x^2, q)\}
\]

\[
= \gamma_A(x^2, q) = \gamma_A(xx, q) \leq \max\{\gamma_A(x, q), \gamma_A(x, q)\} = \gamma_A(x, q),
\]

so that \( \mu_A(x, q) = \mu_A(x^2, q) \) and \( \gamma_A(x, q) = \gamma_A(x^2, q) \). \( \blacksquare \)
Lemma 3.8 ([4]) An ordered semigroup $S$ is left (resp. right) simple if and only if $(Sa) = S$ (resp. $(aS) = S$) for every $a \in S$.

Theorem 3.9 Let $S$ be an ordered semigroup. If $S$ is regular, left and right simple, then every intuitionistic $Q$-fuzzy bi-ideal of $S$ is constant.

Proof. Assume that $S$ is regular, left and right simple. Let $A = \langle \mu_A; \gamma_A \rangle$ be an intuitionistic $Q$-fuzzy bi-ideal of $S$. Consider the set

$$\Omega_S := \{ e \in S \mid e \leq e^2 \}.$$ 

Since $S$ is regular, for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. For the element $ax \in S$, we have $ax \leq (axa)x = (ax)^2$, and so $ax \in \Omega_S$. This means that $\Omega_S \neq \emptyset$. Let $b \in \Omega_S, q \in Q$. Since $b \in S$, we have $\mu_A(b, q) \in [0; 1]$ and $\gamma_A(b, q) \in [0; 1]$. We first show that $\mu_A(b, q) = \mu_A(e, q)$ and $\gamma_A(b, q) = \gamma_A(e, q)$ for all $e \in \Omega_S$. Let $e \in \Omega_S$. Since $S$ is left and right simple, it follows from Lemma 3.8 that $(Sb) = S$ and $(bS) = S$. Since $e \in S$, we have $e \in (Sb)$ and $e \in (bS)$. Thus $e \leq bx$ and $e \leq yb$ for some $x, y \in S$, which imply that $e^2 \leq (bx)e \leq (bx)(yb) = b(xy)b$. Since $A = \langle \mu_A; \gamma_A \rangle$ is an intuitionistic $Q$-fuzzy bi-ideal, we obtain (1)

$$\mu_A(e^2, q) \geq \mu_A(b(xy)b, q) \geq \min\{\mu_A(b, q); \mu_A(b, q)\} = \mu_A(b, q)$$

$$\gamma_A(e^2, q) \leq \gamma_A(b(xy)b, q) \leq \max\{\gamma_A(b, q); \gamma_A(b, q)\} = \gamma_A(b, q).$$

Since $e^2 \in \Omega_S$, we get $e \leq e^2$, and so $\mu_A(e^2, q) \leq \mu_A(e, q)$ and $\gamma_A(e^2, q) \geq \gamma_A(e, q)$. It follows from (1) that $\mu_A(e, q) \geq \mu_A(b, q)$ and $\gamma_A(e, q) \leq \gamma_A(b, q)$. This shows that $A = \langle \mu_A; \gamma_A \rangle$ is constant on $\Omega_S$. Now let $a \in S$. Then $a \leq axa$ for some $x \in S$ since $S$ is regular. It follows that $ax \leq (axa)x = (ax)^2$ and $xa \leq x(axa) = (xa)^2$ so that $ax, xa \in \Omega_S$. Thus, by the previous arguments, we have $\mu_A(ax, q) = \mu_A(b, q) = \mu_A(xa, q)$ and $\mu_A(ax, q) = \mu_A(b, q) = \mu_A(xa, q)$.

Since $(axa)xa = (axa)xa \leq a, a \leq a$, we get

$$\mu_A(a, q) \geq \mu_A((axa)xa, q) \geq \min\{\mu_A(ax, q), \mu_A(xa, q)\}$$

and

$$\gamma_A(a, q) \leq \gamma_A((axa)xa, q) \leq \max\{\gamma_A(ax, q), \gamma_A(xa, q)\}$$

Note that $b \in (Sa)$ and $b \in (aS)$ and hence $b \leq xa$ and $b \leq ay$ for some $x, y \in S$. Hence $b^2 \leq a(yxa)a$ which implies

$$\mu_A(b^2, q) \geq \mu_A(a(yxa)a, q) \geq \min\{\mu_A(a, q), \mu_A(a, q)\} = \mu_A(a, q)$$

and

$$\gamma_A(b^2, q) \leq \gamma_A(a(yxa)a, q) \leq \max\{\gamma_A(a, q), \gamma_A(a, q)\} = \gamma_A(a, q).$$

Since $b \in \Omega_S$, we have $b \leq b^2$. It follows that $\mu_A(b, q) \geq \mu_A(b^2, q) \geq \mu_A(a, q)$ and $\gamma_A(b, q) \leq \gamma_A(b^2, q) \leq \gamma_A(a, q)$. Hence $A = \langle \mu_A; \gamma_A \rangle$ is constant on $S$. 


References


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