On \( Q \)-Fuzzy Bi-\( \Gamma \)-Ideals in \( \Gamma \)-Semigroups

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Abstract
In this paper, we consider the \( Q \)-fuzzification of bi-\( \Gamma \)-ideals in \( \Gamma \)-semigroups, and investigate some of their related properties.

Mathematics Subject Classification: 08A72, 20M12, 20M99, 20N25

Keywords: \( Q \)-fuzzy set, \( \Gamma \)-semigroup, \( Q \)-Fuzzy bi-\( \Gamma \)-ideals

1 Introduction
The concept of fuzzy sets was introduced by Lofti Zadeh[13] in his classic paper in 1965. Azirel Rosenfeld[8] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. The idea of fuzzy subsemigroup was also introduced by Kuroki [3, 5, 6]. In [4], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. The notion of a \( \Gamma \)-semigroup was introduced by Sen and Saha[12] as a generalization of semigroups and ternary semigroup. S.K. Sardar and S.K. Majumder [1, 2, 10, 11] have introduced the notion of fuzzification of ideals, prime ideals, semiprime ideals and ideal extensions of \( \Gamma \)-semigroups and studied them via its operator semigroups. In this paper, we consider a \( Q \)-fuzzification of the concept of a bi-\( \Gamma \)-ideal in a \( \Gamma \)-semigroup, and some properties of such bi-\( \Gamma \)-ideals are investigated.

2 Preliminary Notes
In this section we discuss some elementary definitions that we use in the sequel.

Definition 2.1 Let \( S \) and \( \Gamma \) be two non-empty sets. \( S \) is called a \( \Gamma \)-semigroup if there exist mapping from \( S \times \Gamma \times S \) to \( S \), written as \( (a, \alpha, b) \rightarrow a\alpha b \) satisfying the following associative law: \( (a\alpha b)\beta c = a\alpha (b\beta c) \) for all \( a, b, c \in S \) and for all \( \alpha, \beta \in \Gamma \).
Let $S$ be a $\Gamma$-semigroup. A non-empty subset $A$ of $S$ is said to be a sub $\Gamma$-semigroup of $S$ if $ASA \subseteq A$.

**Definition 2.2** Let $S$ be a $\Gamma$-semigroup. By a left(right) ideal of $S$ we mean a non-empty set $A$ of $S$ such that $SA \subseteq A (AS \subseteq A)$. By two side ideal or simply an ideal, we mean a non-empty subset of $S$ which is both a left and right ideal of $S$.

**Definition 2.3** Let $S$ be a $\Gamma$-semigroup. A sub $\Gamma$-semigroup $A$ of $S$ is called a bi-$\Gamma$-ideal of $S$ if $A \Gamma A \subseteq A$.

Let $Q$ and $X$ be two non-empty sets. A mapping $\mu : X \times Q \to [0, 1]$ is called the $Q$-fuzzy subset of $X$ and the complement of a set $\mu$, denoted by $\mu'$ is the $Q$-fuzzy subset in $X$ given by $\mu'(x, q) = 1 - \mu(x, q)$ for all $x \in X$ and for all $q \in Q$.

Let $\mu$ be a $Q$-fuzzy subset of a non-empty set $X$. Then the set $\mu_t = \{x \in X \mid \mu(x, q) \geq t, \forall q \in Q\}$ for $t \in [0, 1]$, is called the level subset or $t$-level subset of $\mu$.

Let $A$ be a non-empty subset of $X$. Then $\mu_A : X \times Q \to [0, 1]$ is called a characteristic function which is defined by

$$
\mu_A(x, q) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A,
\end{cases}
$$

for all $x \in S$ and for all $q \in Q$.

**Definition 2.4** Let $\mu$ be a $Q$-fuzzy subset of a $\Gamma$-semigroup $S$. A $Q$-fuzzy subset $\mu$ is called a $Q$-fuzzy sub $\Gamma$-semigroup of $S$ if $\mu(x\gamma y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$ for all $x, y \in S, \gamma \in \Gamma$ and $q \in Q$.

### 3 Main Results

In what follow, $S$ will denote a $\Gamma$-semigroup unless otherwise specified. Now, we introduce a notion of $Q$-fuzzy bi-$\Gamma$-ideal of $S$.

**Definition 3.1** A $Q$-fuzzy subset $\mu$ of $S$ is called a $Q$-fuzzy bi-$\Gamma$-ideal of $S$ if

(i) $(\forall x, y \in S, \gamma \in \Gamma, q \in Q)(\mu(x\gamma y, q) \geq \min\{\mu(x, q), \mu(y, q)\})$,

(ii) $(\forall x, y, z \in S, \alpha, \beta \in \Gamma, q \in Q)(\mu(x\alpha y\beta z, q) \geq \min\{\mu(x, q), \mu(z, q)\})$.

**Lemma 3.2** If $B$ is a bi-$\Gamma$-ideal of $S$ then for any $0 < t < 1$, there exists a $Q$-fuzzy bi-$\Gamma$-ideal $\mu$ of $S$ such that $\mu_t = B$
Proof. Let $\mu : S \times Q \to [0, 1]$ be defined by

$$
\mu(x, q) = \begin{cases} 
    t & \text{if } x \in B \\
    0 & \text{if } x \notin B 
\end{cases}
$$

for all $x \in S$ and for all $q \in Q$, where $t$ is a fixed number in $(0, 1)$. Then, clearly $\mu_t = B$.

Now suppose that $B$ is a bi-$\Gamma$-ideal of $S$. For all $x, y, z \in S, q \in Q$ and $\gamma \in \Gamma$ such that $x \gamma y \in B$, we have

$$
\mu(x \gamma y, q) \geq t = \min\{\mu(x, q), \mu(y, q)\}.
$$

Also, for all $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$ such that $x \alpha y \beta z \in B$, we have

$$
\mu(x \alpha y \beta z, q) \geq t = \min\{\mu(x, q), \mu(y, q)\}.
$$

Thus $\mu$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$.

Lemma 3.3 Let $B$ be a non-empty subset of $S$. Then $B$ is a bi-$\Gamma$-ideal of $S$ if and only if $\chi_B$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$.

Proof. Let $x, y \in S, q \in Q$ and $\gamma \in \Gamma$. From the hypothesis, $x \gamma y \in B$.

(i) If $x, y \in B$, then $\chi_B(x, q) = 1$ and $\chi_B(y, q) = 1$. In this case $\chi_B(x \gamma y, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(y, q)\}$.

(ii) If $x \in B$ and $y \notin B$, then $\chi_B(x, q) = 1$ and $\chi(y, q) = 0$. Thus, $\chi_B(x \gamma y, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(y, q)\}$.

(iii) If $x \notin B$ and $y \in B$, then, similarly with case (ii).

(iv) If $x \notin B$ and $y \notin B$, then $\chi_B(x, q) = 0$ and $\chi(y, q) = 0$. Thus, $\chi_B(x \gamma y, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(y, q)\}$.

Let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$. From the hypothesis, $x \alpha y \beta z \in B$.

(i) If $x, z \in B$, then $\chi_B(x, q) = 1$ and $\chi_B(z, q) = 1$. Thus $\chi_B(x \alpha y \beta z, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(z, q)\}$.

(ii) If $x \in B$ and $z \notin B$, then $\chi_B(x, q) = 1$ and $\chi_B(z, q) = 0$. Thus $\chi_B(x \alpha y \beta z, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(z, q)\}$.

(ii) If $x \notin B$ and $z \in B$, then $\chi_B(x, q) = 0$ and $\chi_B(z, q) = 1$. Thus $\chi_B(x \alpha y \beta z, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(z, q)\}$.

(ii) If $x \notin B$ and $z \notin B$, then $\chi_B(x, q) = 0$ and $\chi_B(z, q) = 0$. Thus $\chi_B(x \alpha y \beta z, q) = 1 \geq \min\{\chi_B(x, q), \chi_B(z, q)\}$.

Conversely, suppose $\chi_B$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$. Then by Lemma 3.2, $\chi_B$ is two-value. Hence $B$ is a bi-$\Gamma$-ideal of $S$. This completes the proof.
The following theorem proves that an intersection of $Q$-fuzzy bi-$\Gamma$-ideals is also a $Q$-fuzzy bi-$Q$-ideal.

**Theorem 3.4** If $\{A_i\}_{i \in \lambda}$ is a family of $Q$-fuzzy bi-$\Gamma$-ideals of $S$ then $\cap A_i$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$, where $\cap A_i = \{\cap \mu_i\}$ and $\cap \mu_i(x, q) = \min\{\mu_i(x, q) \mid i \in \lambda\}$ for every $x \in S$ and $q \in Q$.

**Proof.** Let $x, y \in S, q \in Q$. Then we have

(i) $\cap \mu_i(x\gamma y, q) = \inf\{\min\{\mu_i(x, q), \mu_i(y, q)\} \mid i \in \lambda\}$

= $\min\{\{\inf(\mu_i(x, q)), \inf(\mu_i(y, q))\} \mid i \in \lambda\}$

= $\min\{\{\inf(\mu_i(x, q)) \mid i \in \lambda\}, \{\inf(\mu_i(y, q)) \mid i \in \lambda\}\}$

= $\min\{\inf(\mu_i(x, q), \inf(\mu_i(y, q))\}$.

Let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$.

(ii) $\cap \mu_i(x\alpha y\beta z, q) = \inf\{\min\{\mu_i(x, q), \mu_i(z, q)\} \mid i \in \lambda\}$

= $\min\{\{\inf(\mu_i(x, q)), \inf(\mu_i(z, q))\} \mid i \in \lambda\}$

= $\min\{\{\inf(\mu_i(x, q)) \mid i \in \lambda\}, \{\inf(\mu_i(z, q)) \mid i \in \lambda\}\}$

= $\min\{\inf(\mu_i(x, q), \inf(\mu_i(z, q))\}$.

Hence, $\cap A_i$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$. $\blacksquare$

**Theorem 3.5** If $\mu$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$ then $\mu'$ is also a $Q$-fuzzy bi-$\Gamma$-ideal of $M$.

**Proof.** (i) Let $x, y \in S, q \in Q$ and $\gamma \in \Gamma$. We have:

$\mu'(x\gamma y, q) = 1 - \mu(x\gamma y, q)$

= $1 - \min\{\mu(x, q), \mu(y, q)\}$

= $\min\{1 - \mu(x, q), 1 - \mu(y, q)\}$

= $\min\{\mu'(x, q), \mu'(y, q)\}$.

Let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$. We have:

$\mu'(x\alpha y\beta z, q) = 1 - \mu(x\alpha y\beta z, q)$

= $1 - \min\{\mu(x, q), \mu(z, q)\}$

= $\min\{1 - \mu(x, q), 1 - \mu(z, q)\}$

= $\min\{\mu'(x, q), \mu'(z, q)\}$.

Therefore, $\mu'$ is also a $Q$-fuzzy bi-$\Gamma$-ideal of $S$. $\blacksquare$

The following theorem gives the relation between $Q$-fuzzy bi-$\Gamma$-ideal and bi-$\Gamma$-ideal.

**Theorem 3.6** A $Q$-fuzzy subset $\mu$ in a $\Gamma$-ideal of $S$ is a $Q$-fuzzy bi-$\Gamma$-ideal of $S$ if and only if the level set $U(\mu; t) = \{x \in S \mid \mu(x, q) \geq t, \forall q \in Q\}$ is a bi-$\Gamma$-ideal of $S$ when it is non-empty.

**Proof.** Let $\mu$ be a $Q$-fuzzy bi-$\Gamma$-ideal of $S$ and $x, y \in S, q \in Q$. Then $\mu(x\gamma y, q) \geq \min\{\mu(x, q), \mu(y, q)\}$. 

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\[ x, y \in U(\mu;t), q \in Q, \gamma \in \Gamma \Rightarrow \mu(x,q) \geq t, \mu(y,q) \geq t \]
\[ \mu(x\gamma y,q) \geq \min\{\mu(x,q), \mu(y,q)\} \]
\[ \mu(x\gamma y,q) \geq t \]
\[ \Rightarrow x\gamma y \in U(\mu;t). \]

Also, let \( x, y, z \in S, q \in Q \) and \( \alpha, \beta \in \Gamma \). Then
\[ \mu(x\alpha y\beta z) \geq \min\{\mu(x,q), \mu(z,q)\}. \]

Thus, \( U(\mu;t) \) is a bi-Γ-ideal of \( S \).

Conversely, assume that \( U(\mu;t) \) is a bi-Γ-ideal of \( S \).

Let \( t = \min\{\mu(x,q), \mu(y,q)\} \). Then
\[ x, y \in U(\mu;t), q \in Q, \gamma \in \Gamma \Rightarrow x\gamma y \in U(\mu;t) \]
\[ \Rightarrow \mu(x\gamma y,q) \geq t \]
\[ \Rightarrow \mu(x\gamma y,q) \geq \min\{\mu(x,q), \mu(y,q)\}. \]

Next, define \( t = \min\{\mu(x,q), \mu(z,q)\} \). Then
\[ x, y, z \in U(\mu;t), q \in Q, \alpha, \beta \in \Gamma \Rightarrow x\alpha y\beta z \in U(\mu;t) \]
\[ \Rightarrow \mu(x\alpha y\beta z,q) \geq t \]
\[ \Rightarrow \mu(x\alpha y\beta z,q) \geq \min\{\mu(x,q), \mu(z,q)\}. \]

Hence \( \mu \) is a Q-fuzzy bi-Γ-ideal of \( S \).

References


Received: August, 2011