The Relation of Boundary and Exterior Sets in Biminimal Structure Spaces

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Abstract

The purpose of this paper is to introduce the concept and some relation of boundary and exterior sets in biminimal structure spaces.

Keywords: biminimal structure spaces, boundary and exterior sets

1 Introduction

The notion of biminimal structure space was introduced by C. Boonpok [1] in 2010. Also he studied \( m_X^1m_X^2 \)-closed sets and \( m_X^1m_X^2 \)-open sets in biminimal structure spaces. The concept of boundary set in biminimal structure space was introduced by S. Sompong and S. Muangchan [5]. And the notion of exterior set in biminimal structure space was introduced by S. Sompong [4]. In this paper, we introduced the concept and some relation of boundary and exterior sets in biminimal structure spaces.

2 Preliminaries

In this section we recall the notions, notations and some previously results.
**Definition 2.1.** [2] Let $X$ be a nonempty set and $P(X)$ be the power set of $X$. A subfamily $m_X$ of $P(X)$ is called a minimal structure (briefly $m$-structure) on $X$ if $\emptyset \in m_X$ and $X \in m_X$.

By $(X, m_X)$, we denote a nonempty set $X$ with an $m$-structure $m_X$ on $X$ and it is called an $m$-space.

**Definition 2.2.** [1] Let $X$ be a nonempty set and $m^1_X$, $m^2_X$ be minimal structures on $X$. A triple $(X, m^1_X, m^2_X)$ is called a biminimal structure space (briefly bim-space).

We defined all of elements in $m^1_X$ and $m^2_X$ are open sets.

**Definition 2.3.** [1] A subset of a biminimal structure space $(X, m^1_X, m^2_X)$ is called $m^1_Xm^2_X$-closed if $A = m^1\text{Cl}(m^2\text{Cl}(A))$. The complement of $m^1_Xm^2_X$-closed set is called $m^1_Xm^2_X$-open.

**Proposition 2.4.** [1] Let $(X, m^1_X, m^2_X)$ be a biminimal structure space. Then $A$ is $m^1_Xm^2_X$-open subset of $(X, m^1_X, m^2_X)$ if and only if $A = m^1\text{Int}(m^2\text{Int}(A))$.

**Proposition 2.5.** [1] Let $(X, m^1_X, m^2_X)$ be a biminimal structure space. If $A$ and $B$ are $m^1_Xm^2_X$-closed subsets of $(X, m^1_X, m^2_X)$, then $A \cap B$ is $m^1_Xm^2_X$-closed.

**Proposition 2.6.** [1] Let $(X, m^1_X, m^2_X)$ be a biminimal structure space. If $A$ and $B$ are $m^1_Xm^2_X$-open subsets of $(X, m^1_X, m^2_X)$, then $A \cup B$ is $m^1_Xm^2_X$-open.

**Definition 2.7.** [5] Let $(X, m^1_X, m^2_X)$ be a biminimal structure space, $A$ be a subset of $X$ and $x \in X$. We called $x$ is $(i,j) - m_X$ – boundary point of $A$ if $x \in m^i\text{Cl}(m^j\text{Cl}(A)) \cap m^i\text{Cl}(m^j\text{Cl}(X \setminus A))$. We denote the set of all $(i,j) - m_X$ – boundary point of $A$ by $mBdr_{ij}(A)$ where $i, j = 1, 2$ and $i \neq j$.

From definition, we have $mBdr_{ij}(A) = m^i\text{Cl}(m^j\text{Cl}(A)) \cap m^i\text{Cl}(m^j\text{Cl}(X \setminus A))$.

**Definition 2.8.** [4] Let $(X, m^1_X, m^2_X)$ be a biminimal structure space, $A$ be a subset of $X$ and $x \in X$. We called $x$ is $m^1_Xm^2_X$-exterior point of $A$ if $x \in m^i\text{Int}(m^j\text{Int}(X \setminus A))$. We denote the set of all $m^1_Xm^2_X$-exterior point of $A$ by $mExt_{ij}(A)$ where $i, j = 1, 2$ and $i \neq j$.

From above definition, it is easy to see that $mExt_{ij}(A) = X \setminus m^i\text{Cl}(m^j\text{Cl}(A))$.

### 3 The Relation of Boundary and Exterior Sets

In this section, we introduced the concept and some relation of boundary and exterior sets in biminimal structure spaces.
Theorem 3.1. Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A\) be a subset of \(X\). Then for any \(i, j = 1, 2\) and \(i \neq j\), we have
\[m_{Ext_{ij}}(A) \cap m_{Bdr_{ij}}(A) = \emptyset.\]

Proof. \[m_{Ext_{ij}}(A) \cap m_{Bdr_{ij}}(A) = (X \setminus m^1 Cl(m^j Cl(A))) \cap [m^i Cl(m^j Cl(A)) \cap m^j Cl(m^i Cl(X \setminus A))] = \emptyset.\]

Corollary 3.2. Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A\) be a subset of \(X\). Then for any \(i, j = 1, 2\) and \(i \neq j\), we have
\[m_{Ext_{ij}}(A) \cap m_{Bdr_{ij}}(X \setminus A) = \emptyset.\]

Proof. By Lemma 3.3 \([5]\), \(m_{Bdr_{ij}}(X \setminus A) = m_{Bdr_{ij}}(A)\) and Theorem 3.1, we have \(m_{Ext_{ij}}(A) \cap m_{Bdr_{ij}}(X \setminus A) = \emptyset\) for any \(i, j = 1, 2\) and \(i \neq j\).

Example 3.3. Let \(X = \{1, 2, 3\}\). Define \(m\)-structures \(m_X^1\) and \(m_X^2\) on \(X\) as follows: \(m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}\) and \(m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}\).

Hence \(m_{Ext_{12}}(\{2\}) = X \setminus m^1 Cl(m^2 Cl(\{2\})) = \{1\}\),
\[m_{Ext_{21}}(\{2\}) = X \setminus m^2 Cl(m^1 Cl(\{2\})) = \emptyset,\]
\[m_{Bdr_{ij}}(\{2\}) = m^i Cl(m^j Cl(\{2\})) \cap m^j Cl(m^i Cl(X \setminus \{2\})) = \emptyset,\]
and \(m_{Bdr_{21}}(\{2\}) = X\).

Therefore \(m_{Ext_{ij}}(\{2\}) \cap m_{Bdr_{ij}}(\{2\}) = \emptyset\) for any \(i, j = 1, 2\) and \(i \neq j\).

Theorem 3.4. Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A\) be a subset of \(X\). Then we have \(m_{Ext_{ij}} \cap m^j Int(m^i Int(A)) = \emptyset\) for any \(i, j = 1, 2\) and \(i \neq j\).

Proof. \[m_{Ext_{ij}} \cap m^j Int(m^i Int(A)) = (X \setminus m^i Cl(m^j Cl(A))) \cap m^j Int(m^i Int(A)) = m^i Int(m^j Int(X \setminus A)) \cap m^i Int(m^j Int(A)) = \emptyset.\]

Example 3.5. Let \(X = \{1, 2, 3\}\). Define \(m\)-structures \(m_X^1\) and \(m_X^2\) on \(X\) as follows: \(m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}\) and \(m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}\).

Hence \(m_{Ext_{12}}(\{2\}) = X \setminus m^1 Cl(m^2 Cl(\{2\})) = \{1\}\),
\[m_{Ext_{21}}(\{2\}) = X \setminus m^2 Cl(m^1 Cl(\{2\})) = \emptyset,\]
\[m^1 Int(m^2 Int(\{2\})) = \emptyset\] and \(m^2 Int(m^1 Int(\{2\})) = \emptyset\).

Therefore \(m_{Ext_{ij}}(\{2\}) \cap m^j Int(m^i Int(\{2\})) = \emptyset\) for any \(i, j = 1, 2\) and \(i \neq j\).

Theorem 3.6. \([5]\) Let \((X, m_X^1, m_X^2)\) be a biminimal structure space and \(A\) be a subset of \(X\). Then for any \(i, j = 1, 2\) and \(i \neq j\),
\[m_{Bdr_{ij}}(A) \cap m^j Int(m^i Int(A)) = \emptyset.\]
Example 3.7. Let $X = \{1, 2, 3\}$. Define $m$-structures $m_X^1$ and $m_X^2$ on $X$ as follows: $m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$.

Hence $m_{Bdr_{ij}}(\{2\}) = m^i\text{Cl}(m^j\text{Cl}(\{2\})) \cap m^j\text{Cl}(m^i\text{Cl}(X \setminus \{2\}))$.

$m_{Bdr_{12}}(\{2\}) = \{2, 3\}, m_{Bdr_{21}}(\{2\}) = X$.

$m^1\text{Int}(m^2\text{Int}(\{2\})) = \emptyset$ and $m^2\text{Int}(m^1\text{Int}(\{2\})) = \emptyset$.

Therefore $m_{Bdr_{ij}}(\{2\}) \cap m^i\text{Int}(m^j\text{Int}(\{2\})) = \emptyset$ for any $i, j = 1, 2$ and $i \neq j$.

Theorem 3.8. Let $(X, m_X^1, m_X^2)$ be a biminimal structure space and $A$ be a subset of $X$. Then $X = m_{Bdr_{ij}}(A) \cup m_{Ext_{ij}}(A) \cup m^i\text{Int}(m^j\text{Int}(A))$ is a pairwise disjoint union for any $i, j = 1, 2$ and $i \neq j$.

Proof. $m_{Bdr_{ij}}(A) \cup m_{Ext_{ij}}(A) \cup m^i\text{Int}(m^j\text{Int}(A))$

$= [m^i\text{Cl}(m^j\text{Cl}(A)) \setminus m^j\text{Int}(m^i\text{Int}(A))] \cup [X \setminus m^i\text{Cl}(m^j\text{Cl}(A))]

\cup m^j\text{Int}(m^i\text{Int}(A))$

$= [m^i\text{Cl}(m^j\text{Cl}(A)) \setminus m^j\text{Int}(m^i\text{Int}(A))] \cup m^j\text{Int}(m^i\text{Int}(A))

\cup [X \setminus m^i\text{Cl}(m^j\text{Cl}(A))$

$= m^i\text{Cl}(m^j\text{Cl}(A)) \cup [X \setminus m^i\text{Cl}(m^j\text{Cl}(A))]

= X$.

By Theorem 3.1, 3.4 and 3.6, we have $m_{Ext_{ij}}(A) \cap m_{Bdr_{ij}}(A) = \emptyset$, $m_{Ext_{ij}} \cap m^i\text{Int}(m^j\text{Int}(A)) = \emptyset$ and $m_{Bdr_{ij}}(A) \cap m^i\text{Int}(m^j\text{Int}(A)) = \emptyset$.

Therefore $X = m_{Bdr_{ij}}(A) \cup m_{Ext_{ij}}(A) \cup m^i\text{Int}(m^j\text{Int}(A))$ is a pairwise disjoint union for any $i, j = 1, 2$ and $i \neq j$.

Example 3.9. Let $X = \{1, 2, 3\}$. Define $m$-structures $m_X^1$ and $m_X^2$ on $X$ as follows: $m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}$ and $m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}$.

Hence $m_{Bdr_{12}}(\{2\}) = \{2, 3\}, m_{Bdr_{21}}(\{2\}) = X$.

$m_{Ext_{12}}(\{2\}) = \{1\}, m_{Ext_{21}}(\{2\}) = \emptyset$.

$m^1\text{Int}(m^2\text{Int}(\{2\})) = \emptyset$ and $m^2\text{Int}(m^1\text{Int}(\{2\})) = \emptyset$.

Therefore $X = m_{Bdr_{ij}}(A) \cup m_{Ext_{ij}}(A) \cup m^i\text{Int}(m^j\text{Int}(A))$ for any $i, j = 1, 2$ and $i \neq j$.

Theorem 3.10. Let $(X, m_X^1, m_X^2)$ be a biminimal structure space and $A$ be a subset of $X$. If $A$ is $m_X^1 m_X^2$-closed, then for any $i, j = 1, 2$ and $i \neq j$, we have $m_{Ext}(X \setminus m_{Ext}(A)) \cap m_{Bdr}(A) = \emptyset$.

Proof. Since $A$ is $m_X^1 m_X^2$-closed, $m_{Ext}(X \setminus m_{Ext}(A)) = m_{Ext}(A)$, (by Theorem 3.7 [4]). It follows that

$m_{Ext}(X \setminus m_{Ext}(A)) \cap m_{Bdr}(A) = m_{Ext}(A) \cap m_{Bdr}(A) = \emptyset$.

Theorem 3.11. Let $(X, m_X^1, m_X^2)$ be a biminimal structure space and $A, B$ be a subset of $X$. If $A \subseteq B$, then $m_{Ext}(B) \subseteq X \setminus m_{Bdr}(A)$, where $i, j = 1, 2$ and $i \neq j$.
Proof. Since $A \subseteq B$ and Theorem 3.4 [4], \( m_{\text{Ext}}_{ij}(B) \subseteq m_{\text{Ext}}_{ij}(A) \).

From \( m_{\text{Ext}}_{ij}(A) \cap m_{\text{Bdr}}_{ij}(A) = \emptyset \), we have \( m_{\text{Ext}}_{ij}(A) \subseteq X \setminus m_{\text{Bdr}}_{ij}(A) \).

Hence \( m_{\text{Ext}}_{ij}(B) \subseteq X \setminus m_{\text{Bdr}}_{ij}(A) \), where \( i, j = 1, 2 \) and \( i \neq j \).

\( \square \)

From Theorem 3.11, if \( A \subseteq B \), we have \( m_{\text{Ext}}_{ij}(B) \subseteq X \setminus m_{\text{Bdr}}_{ij}(A) \), where \( i, j = 1, 2 \) and \( i \neq j \). But the following example show that \( m_{\text{Bdr}}_{ij}(B) \) need not to be a subset of \( X \setminus m_{\text{Ext}}_{ij}(A) \), where \( i, j = 1, 2 \) and \( i \neq j \).

**Example 3.12.** Let \( X = \{1, 2, 3\} \). Define \( m \)-structures \( m_X^1 \) and \( m_X^2 \) on \( X \) as follows:

\[
m_X^1 = \{\emptyset, \{1\}, \{2, 3\}, X\}
\]

\[
m_X^2 = \{\emptyset, \{2\}, \{1, 3\}, X\}
\]

Then \( m_{\text{Ext}}_{12}(\{2, 3\}) = \emptyset, m_{\text{Bdr}}_{12}(\{2\}) = \{2, 3\}, m_{\text{Bdr}}_{12}(\{3\}) = X \). It follows that \( X \setminus m_{\text{Bdr}}_{12}(\{2\}) = \{1\} \) and \( X \setminus m_{\text{Bdr}}_{12}(\{3\}) = \emptyset \).

Hence we have \( m_{\text{Ext}}_{12}(\{2, 3\}) \subseteq X \setminus m_{\text{Bdr}}_{12}(\{2\}) \) and \( m_{\text{Ext}}_{12}(\{2, 3\}) \subseteq X \setminus m_{\text{Bdr}}_{12}(\{3\}) \).

Since \( m_{\text{Ext}}_{12}(\{2\}) = \{1\}, m_{\text{Ext}}_{12}(\{3\}) = \emptyset, m_{\text{Bdr}}_{12}(\{2, 3\}) = X, X \setminus m_{\text{Ext}}_{12}(\{2\}) = \{2, 3\} \) and \( X \setminus m_{\text{Ext}}_{12}(\{3\}) = X \).

Thus we have \( m_{\text{Bdr}}_{12}(\{2, 3\}) \subseteq X \setminus m_{\text{Ext}}_{12}(\{2\}) \),

but \( m_{\text{Bdr}}_{12}(\{2, 3\}) \nsubseteq X \setminus m_{\text{Ext}}_{12}(\{3\}) \).

**References**


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