$\beta(m, \mu)$-Continuous Functions

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Abstract

The purpose of this paper is to introduce the new notions of $\beta(m, \mu)$-continuous functions as functions from an m-space into a generalized topological space. We obtain some characterizations and several properties of functions.

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1 Introduction

In 1983, Abd El-Monsef et al. [5] introduced and investigated $\beta$-open sets and $\beta$-continuity in topological spaces. Recently, T. Noiri and V. Popa [7] have introduced the concepts of weak $\beta$-continuity and almost $\beta$-continuity. In 1997, T. Noiri and A. Nasef [6] have studied fundamental properties of almost $\beta$-continuous functions. In 2002, Á. Császár [1] introduced the notions of generalized neighborhood systems and generalized topological spaces. He also introduced the concepts of continuous functions and associated interior and closure operators on generalized neighborhood systems and generalized topological spaces. In 2009, V. Popa and T. Noiri [9] introduced the notion of minimal structure. Also they introduced the notions of $m$-open sets and $m$-closed sets and characterized those sets using $m$-closure and $m$-interior operators, respectively. In 2010, C. Boonpok introduced and investigated the concepts of almost $(\mu, m)$-continuous functions [2] and weakly $(\mu, m)$-continuous functions [3] as functions from a generalized topological space into an m-space. In this
paper, we study the new notions of $\beta(m, \mu)$-continuous functions as functions from an m-space into a generalized topological space.

2 Preliminaries

We recall some notions and notations of m-spaces. Let $X$ be a nonempty set and $P(X)$ the power set of $X$. A subfamily $m$ of $P(X)$ is called a minimal structure (briefly m-structure) on $X$ if $\emptyset \in m$ and $X \in m$.

By $(X, m)$, we denote a nonempty set of $X$ with a minimal structure $m$ on $X$ and it is called an m-space. Each member of $m$ is said to be m-open and the complement of an m-open set is said to be m-closed.

Definition 2.1. [9] Let $X$ be a nonempty set and $m$ a minimal structure on $X$. For a subset $A$ of $X$, the m-closure of $A$ and the m-interior of $A$ are defined as follows:

1. $m$-Cl($A$) = $\bigcap\{F : A \subseteq F, X - F \in m\}$;
2. $m$-Int($A$) = $\bigcup\{U : U \subseteq A, U \in m\}$.

Lemma 2.2. [4] Let $X$ be a nonempty set and $m$ an m-structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:

1. $m$-Cl($X - A$) = $X - m$-Int($A$) and $m$-Int($X - A$) = $X - m$-Cl($A$);
2. if $X - A \in m$, then $m$-Cl($A$) = $A$ and if $A \in m$, then $m$-Int($A$) = $A$;
3. $m$-Cl($\emptyset$) = $\emptyset$, $m$-Cl($X$) = $X$, $m$-Int($\emptyset$) = $\emptyset$ and $m$-Int($X$) = $X$;
4. if $A \subseteq B$, then $m$-Cl($A$) $\subseteq$ $m$-Cl($B$) and $m$-Int($A$) $\subseteq$ $m$-Int($B$);
5. $A \subseteq m$-Cl($A$) and $m$-Int($A$) $\subseteq$ $A$;
6. $m$-Cl($m$-Cl($A$)) = $m$-Cl($A$) and $m$-Int($m$-Int($A$)) = $m$-Int($A$).

Lemma 2.3. [4] Let $X$ be a nonempty set with a minimal structure $m$ and $A$ a subset of $X$. Then $x \in m$-Cl($A$) if and only if $U \cap A \neq \emptyset$ for every m-open set $U$ containing $x$.

Definition 2.4. [4] An m-structure $m$ on a nonempty set $X$ is said to have property B if the union of any family of subsets belong to $m$ belong to $m$.

Lemma 2.5. [4] Let $X$ be a nonempty set and $m$ an m-structure on $X$ satisfying property B. For a subset $A$ of $X$, the following properties hold:

1. $A \in m$ if and only if $m$-Int($A$) = $A$;
(2) A is \(m\)-closed if and only if \(m\)-Cl\((A) = A\);

(3) \(m\)-Int\((A) \in m\) and \(m\)-Cl\((A)\) is \(m\)-closed.

**Definition 2.6.** [2] A subset \(A\) of a \(m\)-space \((X, m)\) is said to be \(m\)-regular open (resp., \(m\)-semi-open, \(m\)-preopen, \(m\)-\(\alpha\)-open, \(m\)-\(\beta\)-open) if \(A = m\)-Int\((m\)-Cl\((A))\) (resp., \(A \subseteq m\)-Cl\((m\)-Int\((A))\), \(A \subseteq m\)-Int\((m\)-Cl\((m\)-Int\((A))\)), \(A \subseteq m\)-Cl\((m\)-Int\((m\)-Int\((A))\))). The complement of a \(m\)-regular open (resp. \(m\)-semi-open, \(m\)-preopen, \(m\)-\(\alpha\)-open, \(m\)-\(\beta\)-open) set is called \(m\)-regular closed (resp. \(m\)-semi-closed, \(m\)-preclosed, \(m\)-\(\alpha\)-closed, \(m\)-\(\beta\)-closed).

**Lemma 2.7.** [2] Let \((X, m)\) be a \(m\)-space and \(A\) a subset of \(X\). Then

(1) \(A\) is \(m\)-regular closed if and only if \(A = m\)-Cl\((m\)-Int\((A))\);

(2) \(A\) is \(m\)-semi-closed if and only if \(m\)-Int\((m\)-Cl\((A)) \subseteq A\);

(3) \(A\) is \(m\)-preclosed if and only if \(m\)-Cl\((m\)-Int\((A)) \subseteq A\);

(4) \(A\) is \(m\)-\(\beta\)-closed if and only if \(m\)-Int\((m\)-Cl\((m\)-Int\((A))\)) \subseteq A\).

Now, we recall some basic definitions and notations of generalized topological spaces. Let \(Y\) be a nonempty set and \(\mu\) a collection of subsets of \(Y\). Then \(\mu\) is called a generalized topology (briefly GT) [1] on \(Y\) if \(\emptyset \in \mu\) and \(G_i \in \mu\) for \(i \in I \neq \emptyset\) implies \(\cup_{i \in I} G_i \in \mu\). We call the pair \((Y, \mu)\) a generalized topological space (briefly GTS). The elements of \(\mu\) are called \(\mu\)-open sets and the complements are called \(\mu\)-closed sets.

The closure of a subset \(A\) in a generalized topological space \((Y, \mu)\), denoted by \(c_\mu(A)\), is the intersection of \(\mu\)-closed sets including \(A\). And the interior of \(A\), denoted by \(i_\mu(A)\), is the union of \(\mu\)-open sets contained in \(A\).

**Theorem 2.8.** [1] Let \((Y, \mu)\) be a GTS and \(A \subseteq Y\). Then

(1) \(c_\mu(A) = Y - i_\mu(Y - A)\);

(2) \(i_\mu(A) = Y - c_\mu(Y - A)\).

**Proposition 2.9.** [10] Let \((Y, \mu)\) be a GTS and \(A \subseteq Y\). Then

(1) \(x \in i_\mu(A)\) if and only if there exists \(V \in \mu\) such that \(x \in V \subseteq A\);

(2) \(x \in c_\mu(A)\) if and only if \(V \cap A \neq \emptyset\) for every \(\mu\)-open set \(V\) containing \(x\).

**Proposition 2.10.** [10] Let \((Y, \mu)\) be a GTS. For subsets \(A\) and \(B\) of \(Y\), the following properties hold:

(1) \(c_\mu(Y - A) = Y - i_\mu(A)\) and \(i_\mu(Y - A) = Y - c_\mu(A)\);
(2) if \( Y - A \in \mu \), then \( c_\mu(A) = A \) and if \( A \in \mu \), then \( i_\mu(A) = A \);

(3) if \( A \subseteq B \), then \( c_\mu(A) \subseteq c_\mu(B) \) and \( i_\mu(A) \subseteq i_\mu(B) \);

(4) \( A \subseteq c_\mu(A) \) and \( i_\mu(A) \subseteq A \);

(5) \( c_\mu(c_\mu(A)) = c_\mu(A) \) and \( i_\mu(i_\mu(A)) = i_\mu(A) \).

**Definition 2.11.** \([10]\) Let \((Y, \mu)\) be a GST and \( A \subseteq Y \). Then \( A \) is said to be \( \mu\text{-semiopen} \) (resp., \( \mu\text{-preopen}, \mu\text{-regular open}, \mu\text{-}\beta\text{-open}) if \( A \subseteq c_\mu(i_\mu(A)) \) (resp., \( A \subseteq i_\mu(c_\mu(A)), A = i_\mu(c_\mu(A)), A \subseteq c_\mu(i_\mu(c_\mu(A)))) \). The complement of a \( \mu\text{-semiopen} \) (resp., \( \mu\text{-preopen}, \mu\text{-regular open}, \mu\text{-}\beta\text{-open}) set is called \( \mu\text{-semiclosed} \) (resp., \( \mu\text{-preclosed}, \mu\text{-regular closed}, \mu\text{-}\beta\text{-closed})

3 \( \beta(m, \mu)\text{-continuous functions} \)

In this section, we introduce and study \( \beta(m, \mu)\text{-continuous functions} \) and investigate some of their characterizations.

**Definition 3.1.** A function \( f : (X, m) \to (Y, \mu) \) is said to be \( \beta(m, \mu)\text{-continuous} \) at a point \( x \in X \) if for each \( \mu\text{-open} \) set \( V \) containing \( f(x) \), there exists a \( m\beta\text{-open} \) set \( U \) containing \( x \) such that \( f(U) \subseteq V \). A function \( f : (X, m) \to (Y, \mu) \) is said to be \( \beta(m, \mu)\text{-continuous} \) if it has this property at each point \( x \in X \).

**Example 3.2.** Let \( X = \{1, 2\}, m = \{\emptyset, X\} \) and \( Y = \{a, b\}, \mu = \{\emptyset, \{a\}, Y\} \).
Define \( f : (X, m) \to (Y, \mu) \) as follows: \( f(1) = a, f(2) = b \). Then \( f \) is \( \beta(m, \mu)\text{-continuous} \).

**Definition 3.3.** Let \( X \) be a nonempty set with \( m \) an \( m\text{-structure} \) on \( X \). For a subset \( A \) of \( X \), the \( \beta\text{-closure} \) of \( A \), denoted by \( m\text{-Cl}_\beta(A) \), and the \( \beta\text{-interior} \) of \( A \), denoted by \( m\text{-Int}_\beta(A) \), are defined as follows:

1. \( m\text{-Cl}_\beta(A) = \bigcap\{F : A \subseteq F, X - F \text{ is } m\beta\text{-open set}\}; \)
2. \( m\text{-Int}_\beta(A) = \bigcup\{U : U \subseteq A, U \text{ is } m\beta\text{-open set}\}. \)

**Lemma 3.4.** Let \( X \) be a nonempty set and \( m \) an \( m\text{-structure} \) on \( X \). For a subset \( A \) of \( X \), the following properties hold:

1. \( A \) is \( m\beta\text{-open} \) if and only if \( m\text{-Int}_\beta(A) = A \);
2. \( A \) is \( m\beta\text{-closed} \) if and only if \( m\text{-Cl}_\beta(A) = A \);
3. \( m\text{-Int}_\beta(A) \) is \( m\beta\text{-open} \) and \( m\text{-Cl}_\beta(A) \) is \( m\beta\text{-closed} \).

**Theorem 3.5.** For a function \( f : (X, m) \to (Y, \mu) \), the following properties are equivalent:
(1) \( f \) is \( \beta(m, \mu) \)-continuous at \( x \in X \);

(2) \( x \in m\text{-Int}_\beta(f^{-1}(V)) \) for every \( V \in \mu \) containing \( f(x) \);

(3) \( x \in f^{-1}(c_\mu(f(A))) \) for every subset \( A \) of \( X \) with \( x \in m\text{-Cl}_\beta(A) \);

(4) \( x \in f^{-1}(c_\mu(B)) \) for every subset \( B \) of \( Y \) with \( x \in m\text{-Cl}_\beta(f^{-1}(B)) \);

(5) \( x \in m\text{-Int}_\beta(f^{-1}(B)) \) for every subset \( B \) of \( Y \) with \( x \in f^{-1}(i_\mu(B)) \);

(6) \( x \in f^{-1}(D) \) for every \( \mu \)-closed set \( D \) of \( Y \) such that \( x \in m\text{-Cl}_\beta(f^{-1}(D)) \).

Proof. (1) \( \Rightarrow \) (2): Let \( V \) be any \( \mu \)-open subset of \( Y \) containing \( f(x) \). By assumption, there exists an \( m\beta \)-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \), and so \( x \in U \subseteq f^{-1}(V) \). Since \( U \) is an \( m\beta \)-open set, we have \( x \in m\text{-Int}_\beta(f^{-1}(V)) \).

(2) \( \Rightarrow \) (3): Let \( A \) be a subset of \( X \) such that \( x \in m\text{-Cl}_\beta(A) \) and let \( V \) be any \( \mu \)-open subset of \( Y \) containing \( f(x) \). By (2), we have \( x \in m\text{-Int}_\beta(f^{-1}(V)) \). Then there exists an \( m\beta \)-open subset \( U \) of \( X \) containing \( x \) such that \( x \in U \subseteq f^{-1}(V) \). Since \( x \in m\text{-Cl}_\beta(A) \), \( U \cap A \neq \emptyset \). Thus \( \emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A) \). Since \( V \) is \( \mu \)-open containing \( f(x) \), \( f(x) \in c_\mu(f(A)) \) and hence \( x \in f^{-1}(c_\mu(f(A))) \).

(3) \( \Rightarrow \) (4): Let \( B \) be any subset of \( Y \) such that \( x \in m\text{-Cl}_\beta(f^{-1}(B)) \). By (3), \( x \in f^{-1}(c_\mu(f^{-1}(B)))) \subseteq f^{-1}(c_\mu(B)) \). Hence, \( x \in f^{-1}(c_\mu(B)) \).

(4) \( \Rightarrow \) (5): Let \( B \) be any subset of \( Y \) such that \( x \notin m\text{-Int}_\beta(f^{-1}(B)) \). Then \( x \notin X - m\text{-Int}_\beta(f^{-1}(B)) = m\text{-Cl}_\beta(X - f^{-1}(B)) = m\text{-Cl}_\beta(f^{-1}(Y - B)) \). By (4), we have \( x \in f^{-1}(c_\mu(Y - B)) = f^{-1}(Y - i_\mu(B)) = X - f^{-1}(i_\mu(B)) \). Hence, \( x \notin f^{-1}(i_\mu(B)) \).

(5) \( \Rightarrow \) (6): Let \( D \) be a \( \mu \)-closed subset of \( Y \) such that \( x \notin f^{-1}(D) \). Then \( x \notin X - f^{-1}(D) = f^{-1}(Y - D) = f^{-1}(i_\mu(Y - D)) \) because \( Y - D \) is \( \mu \)-open. By (5), \( x \in m\text{-Int}_\beta(f^{-1}(Y - D)) = m\text{-Int}_\beta(X - f^{-1}(D)) = X - m\text{-Cl}_\beta(f^{-1}(D)) \). Hence, \( x \notin m\text{-Cl}_\beta(f^{-1}(D)) \).

(6) \( \Rightarrow \) (2): Let \( x \in X \) and \( V \) be any \( \mu \)-open subset of \( Y \) containing \( f(x) \). Suppose that \( x \notin m\text{-Int}_\beta(f^{-1}(V)) \). Then \( x \in X - m\text{-Int}_\beta(f^{-1}(V)) = m\text{-Cl}_\beta(X - f^{-1}(V)) = m\text{-Cl}_\beta(f^{-1}(Y - V)) \). By (6), \( x \in f^{-1}(Y - V) = X - f^{-1}(V) \). Hence \( x \notin f^{-1}(V) \). This is a contradiction.

(2) \( \Rightarrow \) (1): Let \( V \) be any \( \mu \)-open subset of \( Y \) containing \( f(x) \). By (2), \( x \in m\text{-Int}_\beta(f^{-1}(V)) \) and hence there exists an \( m\beta \)-open set \( U \) containing \( x \) such that \( x \in U \subseteq f^{-1}(V) \). Therefore, \( f(U) \subseteq f(f^{-1}(V)) \subseteq V \) and so \( f \) is \( \beta(m, \mu) \)-continuous at \( x \).

\[ \square \]

**Theorem 3.6.** For a function \( f : (X, m) \to (Y, \mu) \), the following properties are equivalent:

(1) \( f \) is \( \beta(m, \mu) \)-continuous;
(2) \( f^{-1}(V) \) is \( m, \beta \)-open in \( X \) for every \( \mu \)-open set \( V \) of \( Y \);

(3) \( f(\text{\(m\)-Cl}_\beta(A)) \subseteq c_\mu(f(A)) \) for every subset \( A \) of \( X \);

(4) \( \text{\(m\)-Cl}_\beta(f^{-1}(B)) \subseteq f^{-1}(c_\mu(B)) \) for every subset \( B \) of \( Y \);

(5) \( f^{-1}(i_\mu(B)) \subseteq m-\text{Int}_\beta(f^{-1}(B)) \) for every subset \( B \) of \( Y \);

(6) \( f^{-1}(D) \) is \( m, \beta \)-closed in \( X \) for every \( \mu \)-closed set \( D \) of \( Y \).

Proof. (1) \( \Rightarrow \) (2): Let \( V \in \mu \) and let \( x \in f^{-1}(V) \). Since \( f \) is \( \beta(m, \mu) \)-continuous, there exists an \( m, \beta \)-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \). Since \( U \) is \( m, \beta \)-open, we have \( x \in \text{\(m\)-Int}_\beta(f^{-1}(V)) \). Hence \( f^{-1}(V) = m-\text{Int}_\beta(f^{-1}(V)) \), and so \( f^{-1}(V) \) is \( m, \beta \)-open in \( X \).

(2) \( \Rightarrow \) (3): Let \( A \) be any subset of \( X \). Let \( x \in \text{\(m\)-Cl}_\beta(A) \) and \( V \in \mu \) containing \( f(x) \). By (2), we get that \( x \in m-\text{Int}_\beta(f^{-1}(V)) \). Thus there exists an \( m, \beta \)-open subset \( U \) of \( X \) such that \( x \in U \subseteq f^{-1}(V) \). Since \( x \in \text{\(m\)-Cl}_\beta(A) \), \( U \cap A \neq \emptyset \). Then \( \emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A) \). Since \( V \) is \( \mu \)-open containing \( f(x) \), \( f(x) \in c_\mu(f(A)) \), and so \( x \in f^{-1}(c_\mu(f(A))) \). Hence, \( \text{\(m\)-Cl}_\beta(A) \subseteq f^{-1}(c_\mu(f(A))) \). Then \( f(\text{\(m\)-Cl}_\beta(A)) \subseteq c_\mu(f(A)) \).

(3) \( \Rightarrow \) (4): Let \( B \) be any subset of \( Y \). By (3), \( f(\text{\(m\)-Cl}_\beta(f^{-1}(B))) \subseteq c_\mu(f(f^{-1}(B))) \). Hence, \( \text{\(m\)-Cl}_\beta(f^{-1}(B)) \subseteq f^{-1}(c_\mu(B)) \).

(4) \( \Rightarrow \) (5): Let \( B \) be any subset of \( Y \). By (4), we have

\[
X - m-\text{Int}_\beta(f^{-1}(B)) = m-\text{Cl}_\beta(X - f^{-1}(B))
\]

\[
= m-\text{Cl}_\beta(f^{-1}(Y - B))
\]

\[
\subseteq f^{-1}(c_\mu(Y - B))
\]

\[
= f^{-1}(Y - i_\mu(B))
\]

\[
= X - f^{-1}(i_\mu(B)).
\]

Hence, \( f^{-1}(i_\mu(B)) \subseteq m-\text{Int}_\beta(f^{-1}(B)) \).

(5) \( \Rightarrow \) (6): Let \( D \) be any \( \mu \)-closed subset of \( Y \). Then \( Y - D = i_\mu(Y - D) \).

By (5), \( X - f^{-1}(D) = f^{-1}(Y - D) = f^{-1}(i_\mu(Y - D)) \subseteq m-\text{Int}_\beta(f^{-1}(Y - D)) = m-\text{Int}_\beta(X - f^{-1}(D)) \).

Hence, \( m-\text{Cl}_\beta(f^{-1}(D)) \subseteq f^{-1}(D) \).

(6) \( \Rightarrow \) (2): It is obvious.

(2) \( \Rightarrow \) (1): Let \( x \in X \) and let \( V \) be a \( \mu \)-open subset of \( Y \) containing \( f(x) \). By (2), \( x \in m-\text{Int}_\beta(f^{-1}(V)) \) and hence there exists an \( m, \beta \)-open subset \( U \) of \( X \) containing \( x \) such that \( x \in U \subseteq f^{-1}(V) \). Therefore, \( f(U) \subseteq V \), and so \( f \) is \( \beta(m, \mu) \)-continuous at \( x \). \( \square \)

## 4 Almost \( \beta(m, \mu) \)-continuous functions

**Definition 4.1.** A function \( f : (X, m) \to (Y, \mu) \) is said to be almost \( \beta(m, \mu) \)-continuous at a point \( x \in X \) if for each \( \mu \)-open set \( V \) containing \( f(x) \), there
exists an $m$-$\beta$-open set $U$ containing $x$ such that $f(U) \subseteq i_\mu(c_\mu(V))$. A function $f : (X, m) \to (Y, \mu)$ is said to be almost $(m, \mu)$-continuous if it has this property at each point $x \in X$.

**Remark 4.2.** From the above definitions, we have the following implication but the reverse relation may not be true in general:

$$\beta(m, \mu)\text{-continuous} \Rightarrow \text{almost } \beta(m, \mu)\text{-continuous}.$$ 

**Example 4.3.** Let $X = \{1, 2\}$, $m = \{\emptyset, \{2\}, X\}$ and $Y = \{a, b\}$, $\mu = \{\emptyset, \{a\}, Y\}$. Define $f : (X, m) \to (Y, \mu)$ as follows: $f(1) = a$, $f(2) = b$. Then $f$ is almost $\beta(m, \mu)$-continuous but it is not $\beta(m, \mu)$-continuous.

**Theorem 4.4.** For a function $f : (X, m) \to (Y, \mu)$, the following properties are equivalent:

1. $f$ is almost $\beta(m, \mu)$-continuous at $x \in X$;
2. $x \in m\text{-Int}_\beta(f^{-1}(i_\mu(c_\mu(V))))$ for every $\mu$-open set $V$ containing $f(x)$;
3. $x \in m\text{-Int}_\beta(f^{-1}(V))$ for every $\mu$-regular open set $V$ containing $f(x)$;
4. for every $\mu$-regular open set $V$ containing $f(x)$, there exists an $m$-$\beta$-open set $U$ containing $x$ such that $f(U) \subseteq V$.

**Proof.** (1) $\Rightarrow$ (2): Let $V$ be any $\mu$-open subset of $Y$ containing $f(x)$. Then there exists an $m$-$\beta$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq i_\mu(c_\mu(V))$. Thus $x \in U \subseteq f^{-1}(i_\mu(c_\mu(V)))$, and so $x \in m\text{-Int}_\beta(f^{-1}(i_\mu(c_\mu(V))))$.

(2) $\Rightarrow$ (3): Let $V$ be any $\mu$-regular open subset of $Y$ containing $f(x)$. Since $V = i_\mu(c_\mu(V))$ and by (2), we have $x \in m\text{-Int}_\beta(f^{-1}(V))$.

(3) $\Rightarrow$ (4): Let $V$ be any $\mu$-regular open subset of $Y$ containing $f(x)$. By (3), there exists an $m$-$\beta$-open set $U$ containing $x$ such that $U \subseteq f^{-1}(V)$.

(4) $\Rightarrow$ (1): Let $V$ be any $\mu$-open subset of $Y$ containing $f(x)$. Then $f(x) \in V \subseteq i_\mu(c_\mu(V))$. Since $i_\mu(c_\mu(V))$ is $\mu$-regular open, by (4), there exists an $m$-$\beta$-open set $U$ containing $x$ such that $f(U) \subseteq i_\mu(c_\mu(V))$. Hence, $f$ is almost $\beta(m, \mu)$-continuous at $x$.

**Theorem 4.5.** For a function $f : (X, m) \to (Y, \mu)$, the following properties are equivalent:

1. $f$ is almost $\beta(m, \mu)$-continuous;
2. $f^{-1}(V) \subseteq m\text{-Int}_\beta(f^{-1}(i_\mu(c_\mu(V))))$ for every $\mu$-open set $V$ of $Y$;
3. $m\text{-Cl}_\beta(f^{-1}(c_\mu(i_\mu(F)))) \subseteq f^{-1}(F)$ for every $\mu$-closed subset $F$ of $Y$;
4. $m\text{-Cl}_\beta(f^{-1}(c_\mu(i_\mu(c_\mu(B)))))) \subseteq f^{-1}(c_\mu(B))$ for every subset $B$ of $Y$;
(5) \( f^{-1}(i_{\mu}(B)) \subseteq m\text{-Int}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(i_{\mu}(B)))) \) for every subset \( B \) of \( Y \);

(6) \( f^{-1}(V) \) is \( \mu\)-\( \beta \)-open in \( X \) for every \( \mu \)-regular open subset \( V \) of \( Y \);

(7) \( f^{-1}(F) \) is \( \mu\)-\( \beta \)-closed in \( X \) for every \( \mu \)-regular closed subset \( V \) of \( Y \).

Proof. (1) \( \Rightarrow \) (2): Let \( V \) be any \( \mu \)-open subset of \( Y \) and \( x \in f^{-1}(V) \). By (1), there exists an \( \mu\)-\( \beta \)-open subset \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq i_{\mu}(c_{\mu}(V)) \). Thus \( x \in m\text{-Int}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(V)))) \). Then \( f^{-1}(V) \subseteq m\text{-Int}_{\beta}(i_{\mu}(c_{\mu}(V))) \).

(2) \( \Rightarrow \) (3): Let \( F \) be a \( \mu\)-\( \beta \)-closed subset of \( Y \). By (2), we have \( X - f^{-1}(F) = f^{-1}(Y - F) \subseteq m\text{-Int}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(Y - F)))) = m\text{-Int}_{\beta}(f^{-1}(Y - (c_{\mu}(i_{\mu}(F)))) = \) m\text{-Int}_{\beta}(X - f^{-1}(c_{\mu}(i_{\mu}(F)))) = X - m\text{-Cl}_{\beta}(f^{-1}(c_{\mu}(i_{\mu}(F)))) \). This implies that \( m\text{-Cl}_{\beta}(c_{\mu}(i_{\mu}(F))) \subseteq f^{-1}(F) \).

(3) \( \Rightarrow \) (4): Let \( B \) be a subset of \( Y \). Since \( c_{\mu}(B) \) is \( \mu \)-closed and by (3), we have \( m\text{-Cl}_{\beta}(f^{-1}(c_{\mu}(i_{\mu}(c_{\mu}(B)))) \subseteq f^{-1}(c_{\mu}(B)) \).

(4) \( \Rightarrow \) (5): Let \( B \) be any subset of \( Y \). By (4), we have \( X - f^{-1}(c_{\mu}(Y - B)) \subseteq X - m\text{-Cl}_{\beta}(f^{-1}(c_{\mu}(c_{\mu}(Y - B)))) \). Then \( f^{-1}(i_{\mu}(B)) \subseteq m\text{-Int}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(B)))) \).

(5) \( \Rightarrow \) (6): Let \( V \) be any \( \mu \)-regular open subset of \( Y \). Since \( i_{\mu}(c_{\mu}(i_{\mu}(V))) = V \) and by (5), \( f^{-1}(V) \subseteq m\text{-Int}_{\beta}(f^{-1}(V)) \). Then \( f^{-1}(V) = m\text{-Int}_{\beta}(f^{-1}(V)) \).

(6) \( \Rightarrow \) (7): Let \( F \) be any \( \mu \)-regular closed subset of \( Y \). By (6), we have \( X - f^{-1}(F) = f^{-1}(Y - F) = m\text{-Int}_{\beta}(f^{-1}(Y - F)) = X - m\text{-Cl}_{\beta}(f^{-1}(F)) \).

(7) \( \Rightarrow \) (1): Let \( x \in X \) and let \( V \) be any \( \mu \)-regular open subset of \( Y \) containing \( f(x) \). By (7), \( X - f^{-1}(V) = f^{-1}(Y - V) = m\text{-Cl}_{\beta}(f^{-1}(Y - V)) = X - m\text{-Int}_{\beta}(f^{-1}(V)) \). Since \( x \in f^{-1}(V) = m\text{-Int}_{\beta}(f^{-1}(V)) \), there exists an \( \mu\)-\( \beta \)-open set \( U \) containing \( x \) such that \( U \subseteq f^{-1}(V) \). Hence, by Theorem 4.4(4), \( f \) is almost \( \beta(m, \mu) \)-continuous at \( x \). Then \( f \) is almost \( \beta(m, \mu) \)-continuous.

**Theorem 4.6.** For a function \( f : (X, m) \rightarrow (Y, \mu) \), the following properties are equivalent:

(1) \( f \) is almost \( \beta(m, \mu) \)-continuous;

(2) \( m\text{-Cl}_{\beta}(f^{-1}(U)) \subseteq f^{-1}(c_{\mu}(U)) \) for every \( \mu\)-\( \beta \)-open subset \( U \) of \( Y \);

(3) \( m\text{-Cl}_{\beta}(f^{-1}(U)) \subseteq f^{-1}(c_{\mu}(U)) \) for every \( \mu \)-semiopen subset \( U \) of \( Y \);

(4) \( f^{-1}(U) \subseteq m\text{-Int}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(U)))) \) for every \( \mu \)-preopen subset \( U \) of \( Y \).

Proof. (1) \( \Rightarrow \) (2): Let \( U \) be any \( \mu\)-\( \beta \)-open subset of \( Y \). Since \( c_{\mu}(U) \) is \( \mu \)-regular closed, by Theorem 4.5(7), \( m\text{-Cl}_{\beta}(f^{-1}(c_{\mu}(U))) = f^{-1}(c_{\mu}(U)) \). Thus \( m\text{-Cl}_{\beta}(f^{-1}(U)) \subseteq m\text{-Cl}_{\beta}(f^{-1}(c_{\mu}(U))) = f^{-1}(c_{\mu}(U)) \).

(2) \( \Rightarrow \) (3): It follows from the fact that every \( \mu \)-semiopen set is \( \mu\)-\( \beta \)-open.

(3) \( \Rightarrow \) (1): Let \( F \) be any \( \mu \)-regular closed subset of \( Y \). Since \( F \) is \( \mu \)-semiopen, we have \( m\text{-Cl}_{\beta}(f^{-1}(F)) \subseteq f^{-1}(c_{\mu}(F)) = f^{-1}(F) \). Thus, by Theorem 4.5(7), \( f \) is almost \( \beta(m, \mu) \)-continuous.
(1) \Rightarrow (4): Let \( U \) be any \( \mu \)-preopen subset of \( Y \). Then \( U \subseteq i_\mu(c_\mu(U)) \) and \( i_\mu(c_\mu(U)) \) is \( \mu \)-regular open. By Theorem 4.5(6), we have \( f^{-1}(i_\mu(c_\mu(U))) = m-\text{Int}_\beta(f^{-1}(i_\mu(c_\mu(U)))) \). Thus \( f^{-1}(U) \subseteq m-\text{Int}_\beta(f^{-1}(i_\mu(c_\mu(U)))) \).

(4) \Rightarrow (1): Let \( U \) be any \( \mu \)-regular open subset of \( Y \). Then \( U \) is \( \mu \)-preopen and \( f^{-1}(U) \subseteq m-\text{Int}_\beta(f^{-1}(i_\mu(c_\mu(U)))) = m-\text{Int}_\beta(f^{-1}(U)) \). Hence, by Theorem 4.5(6), \( f \) is almost \( \beta(m, \mu) \)-continuous. 

\[ \square \]

5 Weakly \( \beta(m, \mu) \)-continuous functions

Definition 5.1. A function \( f : (X, m) \to (Y, \mu) \) is said to be weakly \( \beta(m, \mu) \)-continuous at a point \( x \in X \) if for each \( \mu \)-open set \( V \) containing \( f(x) \), there exists an \( m-\beta \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq c_\mu(V) \). A function \( f : (X, m) \to (Y, \mu) \) is said to be weakly \( \beta(m, \mu) \)-continuous if it has this property at each point \( x \in X \).

Remark 5.2. From the above definitions, we have a following implication but the reverse relation may not be true in general:

almost \( \beta(m, \mu) \)-continuous \( \Rightarrow \) weakly \( \beta(m, \mu) \)-continuous.

Example 5.3. Let \( X = \{1, 2, 3\} \), \( m = \{\emptyset, \{2\}, \{1, 2\}, X\} \) and \( Y = \{a, b, c\} \), \( \mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\} \). Define \( f : (X, m) \to (Y, \mu) \) as follows: \( f(1) = a \), \( f(2) = c \), \( f(3) = b \). Then \( f \) is weakly \( \beta(m, \mu) \)-continuous but it is not almost \( \beta(m, \mu) \)-continuous.

Theorem 5.4. A function \( f : (X, m) \to (Y, \mu) \) is weakly \( \beta(m, \mu) \)-continuous at \( x \) if and only if for each \( \mu \)-open set \( V \) containing \( f(x) \), \( x \in m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \).

Proof. Assume that \( f \) is weakly \( \beta(m, \mu) \)-continuous at \( x \). Let \( V \) be a \( \mu \)-open set containing \( f(x) \). Thus there exists an \( m-\beta \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq c_\mu(V) \). Then \( x \in U \subseteq f^{-1}(c_\mu(V)) \), and so \( x \in m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \).

Conversely, let \( x \in X \) and \( V \) a \( \mu \)-open set of \( Y \) containing \( f(x) \). By assumption, we have \( x \in m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \). Put \( U = m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \). Then \( U \) is an \( m-\beta \)-open set in \( X \) containing \( x \) such that \( f(U) \subseteq c_\mu(V) \).

Theorem 5.5. A function \( f : (X, m) \to (Y, \mu) \) is weakly \( \beta(m, \mu) \)-continuous if and only if \( f^{-1}(V) \subseteq m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \) for every \( \mu \)-open set \( V \) of \( Y \).

Proof. Assume that \( f \) is weakly \( \beta(m, \mu) \)-continuous. Let \( V \) be a \( \mu \)-open set of \( Y \) and \( x \in f^{-1}(V) \). Then \( f(x) \in V \). Since \( f \) is weakly \( \beta(m, \mu) \)-continuous at \( x \), by Theorem 5.4, \( x \in m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \). Hence, \( f(V) \subseteq m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \).

Conversely, let \( x \in X \) and let \( V \) be a \( \mu \)-open set containing \( f(x) \). By assumption, we have \( x \in f^{-1}(V) \subseteq m-\text{Int}_\beta(f^{-1}(c_\mu(V))) \). By Theorem 5.4, \( f \) is weakly \( \beta(m, \mu) \)-continuous. 

\[ \square \]
Theorem 5.6. For a function $f : (X, m) \to (Y, \mu)$, the following properties are equivalent:

(1) $f$ is weakly $\beta(m, \mu)$-continuous;

(2) $f^{-1}(V) \subseteq m\text{-Int}_{\beta}(f^{-1}(c_{\mu}(V)))$ for every $\mu$-open subset $V$ of $Y$;

(3) $m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(F))) \subseteq f^{-1}(F)$ for every $\mu$-closed subset $F$ of $Y$;

(4) $m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(A))) \subseteq f^{-1}(c_{\mu}(A))$ for every subset $A$ of $Y$;

(5) $f^{-1}(i_{\mu}(A)) \subseteq m\text{-Int}_{\beta}(f^{-1}(c_{\mu}(i_{\mu}(A))))$ for every subset $A$ of $Y$;

(6) $m\text{-Cl}_{\beta}(f^{-1}(V)) \subseteq f^{-1}(c_{\mu}(V))$ for every $\mu$-open subset $V$ of $Y$.

Proof. (1) $\Rightarrow$ (2): It follows from only if part of Theorem 5.5.

(2) $\Rightarrow$ (3): Let $F$ be any $\mu$-closed subset of $Y$. Then $Y - F$ is a $\mu$-open subset of $Y$. By (2), we have $X - f^{-1}(F) = f^{-1}(Y - F) \subseteq m\text{-Int}_{\beta}(f^{-1}(c_{\mu}(Y - F))) = m\text{-Int}_{\beta}(f^{-1}(Y - i_{\mu}(F))) = m\text{-Int}_{\beta}(X - f^{-1}(i_{\mu}(F))) = X - m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(F)))$. Thus $m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(F))) \subseteq f^{-1}(F)$.

(3) $\Rightarrow$ (4): Let $A$ be a subset of $Y$. Since $c_{\mu}(A)$ is $\mu$-closed in $Y$ and by (3), it follows that $m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(A)))) \subseteq f^{-1}(c_{\mu}(A))$.

(4) $\Rightarrow$ (5): Let $A$ be a subset of $Y$. From (4), it follows that $f^{-1}(i_{\mu}(A)) = X - f^{-1}(c_{\mu}(Y - A)) \subseteq X - m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(Y - A)))) = m\text{-Int}_{\beta}(f^{-1}(c_{\mu}(i_{\mu}(A))))$.

(5) $\Rightarrow$ (6): Let $V$ be a $\mu$-open subset of $Y$. Suppose that $x \notin f^{-1}(c_{\mu}(V))$. Then $f(x) \notin c_{\mu}(V)$, and so there exists a $\mu$-open set $W$ containing $f(x)$ such that $V \cap W = \emptyset$. Thus $c_{\mu}(W) \cap V = \emptyset$. By (5), $x \in f^{-1}(W) \subseteq m\text{-Int}_{\beta}(f^{-1}(c_{\mu}(W)))$. Then there exists an $m\text{-}\beta$-open set $G$ containing $x$ such that $x \in G \subseteq f^{-1}(c_{\mu}(W))$. Since $c_{\mu}(W) \cap V = \emptyset$ and $f(G) \subseteq c_{\mu}(W)$, we have $G \cap f^{-1}(V) = \emptyset$. Thus $x \notin m\text{-Cl}_{\beta}(f^{-1}(V))$. Hence $m\text{-Cl}_{\beta}(f^{-1}(V)) \subseteq f^{-1}(c_{\mu}(V))$.

(6) $\Rightarrow$ (1): Let $x \in X$ and let $V$ be a $\mu$-open subset of $Y$ containing $f(x)$. Since $V = i_{\mu}(V) \subseteq i_{\mu}(c_{\mu}(V))$ and by (6), we have $x \in f^{-1}(V) \subseteq f^{-1}(i_{\mu}(c_{\mu}(V))) = X - f^{-1}(c_{\mu}(Y - c_{\mu}(V))) \subseteq X - m\text{-Cl}_{\beta}(f^{-1}(Y - c_{\mu}(V))) = m\text{-Int}_{\beta}(f^{-1}(c_{\mu}(V)))$. Then there exists an $m\text{-}\beta$-open subset $W$ of $X$ containing $x$ such that $W \subseteq f^{-1}(c_{\mu}(V))$. Hence, $f$ is weakly $\beta(m, \mu)$-continuous.

Theorem 5.7. For a function $f : (X, m) \to (Y, \mu)$, the following properties are equivalent:

(1) $f$ is weakly $\beta(m, \mu)$-continuous;

(2) $m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(F))) \subseteq f^{-1}(F)$ for every $\mu$-regular closed subset $F$ of $Y$;

(3) $m\text{-Cl}_{\beta}(f^{-1}(i_{\mu}(c_{\mu}(G)))) \subseteq f^{-1}(c_{\mu}(G))$ for every $\mu$-$\beta$-open subset $G$ of $Y$;
Proof. (1) ⇒ (2): Let $F$ be any $\mu$-regular closed subset of $Y$. Then $i_\mu(F)$ is $\mu$-open, by Theorem 5.6(6), we have $m-\text{Cl}_\beta(f^{-1}(i_\mu(F))) \subseteq f^{-1}(c_\mu(F)))$. Since $F$ is $\mu$-regular closed, we have $m-\text{Cl}_\beta(f^{-1}(i_\mu(F))) \subseteq f^{-1}(c_\mu(i_\mu(F))) = f^{-1}(F)$.

(2) ⇒ (3): Let $G$ be any $\mu$-$\beta$-open subset of $Y$. Then $c_\mu(G) = c_\mu(i_\mu(c_\mu(G)))$, and so $c_\mu(G)$ is $\mu$-regular closed. From (2), we have $m-\text{Cl}_\beta(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$.

(3) ⇒ (4): It follows from the fact that every $\mu$-semiopen set is $\mu$-$\beta$-open.

(4) ⇒ (1): Let $V$ be any $\mu$-open subset of $Y$. Then, by (4), $m-\text{Cl}_\beta(f^{-1}(V)) \subseteq m-\text{Cl}_\beta(f^{-1}(i_\mu(c_\mu(V)))) \subseteq f^{-1}(c_\mu(V))$. Hence, by Theorem 5.6(6), $f$ is weakly $\beta(m,\mu)$-continuous. \qed

Theorem 5.8. For a function $f : (X, m) \to (Y, \mu)$, the following properties are equivalent:

(1) $f$ is weakly $\beta(m,\mu)$-continuous;

(2) $m-\text{Cl}_\beta(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$ for every $\mu$-preopen subset $G$ of $Y$;

(3) $m-\text{Cl}_\beta(f^{-1}(G)) \subseteq f^{-1}(c_\mu(G))$ for every $\mu$-preopen subset $G$ of $Y$;

(4) $f^{-1}(G) \subseteq m-\text{Int}_\beta(f^{-1}(c_\mu(G)))$ for every $\mu$-preopen subset $G$ of $Y$.

Proof. (1) ⇒ (2): Let $G$ be any $\mu$-preopen subset of $Y$. Then $c_\mu(G) = c_\mu(i_\mu(c_\mu(G)))$, and so $c_\mu(G)$ is $\mu$-regular closed. From Theorem 5.7(2), it follows that $m-\text{Cl}_\beta(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$.

(2) ⇒ (3): Let $G$ be any $\mu$-preopen subset of $Y$. Then $G \subseteq i_\mu(c_\mu(G))$, by (2), we have $m-\text{Cl}_\beta(f^{-1}(G)) \subseteq m-\text{Cl}_\beta(f^{-1}(i_\mu(c_\mu(G)))) \subseteq f^{-1}(c_\mu(G))$.

(3) ⇒ (4): Let $G$ be any $\mu$-preopen subset of $Y$. By (3), we have $f^{-1}(G) \subseteq f^{-1}(i_\mu(c_\mu(G))) = X - f^{-1}(Y - i_\mu(c_\mu(G))) = X - f^{-1}(c_\mu(Y - c_\mu(G))) \subseteq X - m-\text{Cl}_\beta(f^{-1}(Y - c_\mu(G))) = m-\text{Int}_\beta(f^{-1}(c_\mu(G))))$.

(4) ⇒ (1): Since every $\mu$-open set is $\mu$-preopen, by (4) and Theorem 5.6(2), it follows that $f$ is weakly $\beta(m,\mu)$-continuous. \qed

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References


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