Positive Periodic Solutions of Singular First Order Functional Difference Equation

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Abstract
In this paper, we consider the following singular first order functional difference equation
\[ \Delta x(k) = -a(k)x(k) + \lambda b(k)f(x(k - \tau(k))), \quad k \in \mathbb{Z} \]
where \( f(.) \) is singular at \( x = 0 \). By using a Kranoselskii fixed point theorem, we will establish the existence and multiplicity of positive periodic solutions for the above problem. The results obtained are new, and some examples are given to illustrate our main results.

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1 Introduction

Consider
\[ \Delta x(k) = -a(k)x(k) + \lambda b(k)f(x(k - \tau(k))), \quad k \in \mathbb{Z} \] (1)
where \( Z \) is the set of integer numbers, \( \omega \in \mathbb{N} \) is a fixed integer, \( \lambda > 0, a : Z \to (0, 1), b : Z \to [0, \infty), \tau : Z \to Z \) are \( \omega \)-periodic and continuous with \( 0 < a(k) < 1 \) for all \( k \in [0, \omega - 1] \) and \( f \in C((\mathbb{R}_+ \setminus \{0\}), (0, \infty)) \).

The existence of positive periodic solutions of functional differential and difference equations have gained the attention of many researches in recent times, see for example [3, 4, 9, 14, 16, 17, 18, 24]. Positive periodic solutions of functional difference equations appears in many ecological models. See for example the discrete Hematopoiesis model [9, 12, 17, 22, 24]

\[
\Delta x(k) = -a(k)x(k) + b(k)e^{-\beta(k)x(k-\tau(k))};
\]

more general the discrete model of blood cell production [6, 9, 12, 13, 17, 24]

\[
\Delta x(k) = -a(k)x(k) + b(k) \frac{1}{1 + x(k - \tau(k))^n}, \quad n > \mathbb{Z}_+;
\]

\[
\Delta x(k) = -a(k)x(k) + b(k) \frac{x(k - \tau(k))}{1 + x(k - \tau(k))^n}, \quad n > \mathbb{Z}_+;
\]

and also the discrete Nicholson’s blowflies model [7, 9, 12, 17, 23, 24]

\[
\Delta x(k) = -a(k)x(k) + b(k)x(k - \tau(k))e^{-\beta(k)x(k-\tau(k))}.
\]

Here \( \Delta \) is the forward difference operator, \( \Delta x(k) = x(k + 1) - x(k) \).

In [18], Wang established the existence, multiplicity and nonexistence of positive periodic solutions for the general scalar of nonlinear nonautonomous functional differential equations. The existence of positive periodic solutions for first order singular boundary value problems have been studied extensively, see [1, 5, 8, 19, 21]. However, to the best of our knowledge, the results concerning functional difference equations are few (see, for example [2, 15, 16]).

In [16], Raffoul proved the existence of positive periodic solutions of scalar nonlinear functional difference equation

\[ x(n + 1) = a(n)x(n) + h(n)f(x(n - \tau(n))), \]

where \( a(n), h(n) \) and \( \tau(n) \) are \( T \)-periodic for \( T \) is an integer with \( T \geq 1 \) under the assumptions that \( a(n), f(x) \) and \( h(n) \) are nonnegative with \( 0 < a(n) < 1 \) for all \( n \in [0, T - 1] \).

In this paper, we extend results in [16] with \( f \) is singular at zero. Our approach is similar to Wang [19] who established the existence and multiplicity of positive periodic solutions for first order non-autonomous singular systems. We shall establish new results on the existence and multiplicity of positive solutions of equation (1) by utilizing the well-known theory of Kranoselskii fixed point theorem. Throughout this paper, we denote the product of \( x(k) \) from \( k = a \) to \( k = b \) by \( \prod_{k=a}^{b} x(k) \) with the understanding that \( \prod_{k=a}^{b} x(k) := 1 \) for all \( a > b \).
2 Preliminaries

In order to simplifying the proof of our main results, we need the following lemmas.

Let \( \mathbb{R}_+ = [0, \infty) \) and \( X \) be the set of all real \( \omega \)-periodic sequences \( x : \mathbb{Z} \to \mathbb{R}_+ \), endowed with the maximum norm

\[
\|x\| = \max_{k \in [0, \omega - 1]} |x(k)|.
\]

Thus \( X \) is a Banach space. We make the following assumptions:

(H1) \( 0 < a(k) < 1, b(k) \) for all \( k \in [0, \omega - 1] \) are continuous and \( \omega \)-periodic such \( a(k) = a(k + \omega), b(k) = b(k + \omega) \) where \( \omega \) is a constant denoting the common period.

(H2) \( f : \mathbb{R}_+ \setminus \{0\} \to (0, \infty) \) is continuous.

We now state the Kranoselskii fixed point theorem [10].

**Lemma 2.1.** Let \( X \) be a Banach space, and let \( K \subset X \) be a cone in \( X \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( X \) with \( 0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2 \), and let

\[
T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K
\]

be a completely continuous operator such that either

(i) \( \|Tx\| \leq \|x\|, x \in K \cap \partial \Omega_1 \) and \( \|Tx\| \geq \|x\|, x \in K \cap \partial \Omega_2 \); or

(ii) \( \|Tx\| \geq \|x\|, x \in K \cap \partial \Omega_1 \) and \( \|Tx\| \leq \|x\|, x \in K \cap \partial \Omega_2 \);

Then \( T \) has a fixed point in \( T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K \).

**Lemma 2.2.** Assume (H1), (H2) hold. If \( x \in X \) then \( x \) is a solution of \( (1) \) if and only if

\[
x(k) = \sum_{s=k}^{k+\omega-1} G(k,s)\lambda b(s)f(x(s - \tau(s))), \quad k \in \mathbb{Z}
\]

where

\[
G(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1 - a(r))}{1 - \prod_{r=0}^{s-1}(1 - a(r))}, \quad s \in [k, k + \omega - 1].
\]
Note that the denominator in $G(k, s)$ is not zero since $0 < a(k) < 1$ for $k \in [0, \omega - 1]$. The construction of $G(k, s)$ is due to Zeng [24]. We remark the process of proof is similar and is omitted.

It is clear that $G(k, s) = G(k + \omega, s + \omega)$ for all $(k, s) \in \mathbb{Z}^2$. A direct calculation shows that

\[ m := \frac{\prod_{r=0}^{\omega-1} (1 - a(r))}{1 - \prod_{r=0}^{\omega-1} (1 - a(r))} \leq G(k, s) \leq \frac{1}{1 - \prod_{r=0}^{\omega-1} (1 - a(r))} =: M. \]

Define $\sigma = \prod_{r=0}^{\omega-1} (1 - a(r))$ satisfying

\[ \sigma \leq G(k, s) \leq \frac{1}{1 - \sigma}, \quad k \leq s \leq k + \omega. \]

Thus, clearly $\sigma = \frac{m}{\lambda} > 0$,

\[ \|x\| = \max_{k \in [0, \omega - 1]} |x(k)| \leq M \sum_{k=0}^{\omega-1} \lambda b(k) f(x(k - \tau(k))). \]

Therefore

\[ x(k) \geq m \lambda \sum_{k=0}^{\omega-1} b(k) f(x(k - \tau(k))) \geq \frac{m}{M} \sum_{k=0}^{\omega-1} b(k) f(x(k - \tau(k))) \geq \sigma \|x\|. \]

Now we define a cone

\[ K = \left\{ x \in X, k \in [0, \omega], x(k) \geq \frac{m}{\lambda} \|x\| = \sigma \|x\| \right\}. \]

It is clear that $K$ is a cone in $X$ and $\min_{k \in [0, \omega]} |x(k)| \geq \sigma \|x\|$ for $x \in K$. For $r > 0$, define $\Omega_r = \{ x \in K : \|x\| < r \}$. Note that $\partial \Omega_r = \{ x \in K : \|x\| = r \}$.

Define a mapping $T : K \setminus \{0\} \to X$ by

\[ Tx(k) = \lambda \sum_{s=k}^{k+\omega-1} G(k, s) b(s) f(x(s - \tau(s))), \quad (3) \]

where $G(k, s)$ is given by (2). By the nonnegativity of $\lambda, f, a, b,$ and $G$, $Tx(k) \geq 0$ on $[0, \omega - 1]$, it is clear that $Tx(k + \omega) = Tx(k)$.

**Lemma 2.3.** $T : K \setminus \{0\} \subset K$ is well-defined.

**Proof.** For any $x \in K \setminus \{0\}$, for all $k \in [0, \omega]$ we have

\[ \|Tx\| = \max_{k \in [0, \omega - 1]} |Tx(k)| \leq M \sum_{s=0}^{\omega-1} \lambda b(s) f(x(s - \tau(s))). \]
Therefore
\[
Tx(k) = \lambda \sum_{s=k}^{k+\omega-1} G(k, s)b(s)f(x(s - \tau(s))) \\
\geq \lambda m \sum_{s=0}^{\omega-1} b(s)f(x(s - \tau(s))) \\
\geq \frac{m}{M} \|Tx\).
\]

Hence \(Tx(k) \geq \sigma \|Tx\) . This implies that \(T : K\{0\} \subset K\) . \(\square\)

**Lemma 2.4.** If (H1) and (H2) hold then the operator \(T : K\{0\} \to K\) is completely continuous.

**Proof.** Let \(x_m(k), x_0(k) \in K\{0\}\) with \(x_m(k) \to x_0(k)\) as \(m \to \infty\). From (3) and since \(f(k, \xi)\) is continuous in \(\xi\), as \(m \to \infty\), we have
\[
|Tx_m(k) - Tx_0(k)| \leq M \sum_{s=0}^{\omega-1} \lambda |b(s)||f(x_m(s)) - f(x_0(s))| \to 0.
\]

Hence \(\|Tx_m - Tx_0\| \to 0\). It follows that the operator \(T\) is continuous.

Further if \(x \subset X\) is a bounded set, then \(\|x\| \leq C_1 = \text{const}\) for all \(x \in K\{0\}\). Set \(C_2 = \max f(k, x(k - \tau(k))), x \in K\{0\}\) then from (3) we get, for all \(x \in K\{0\}\),
\[
\|Tx\| \leq M \sum_{s=k}^{k+\omega-1} \lambda |b(s)||f(k, (x(k - \tau(k))))| \leq M\omega C_2.
\]

This shows that \(T(K\{0\})\) is a bounded set in \(K\). Since \(K\) is n-dimensional, \(T(K\{0\})\) is relatively compact in \(K\). Therefore \(T\) is a completely continuous operator. \(\square\)

For the next following lemmas, we now introduce some notations. For \(r > 0\), let
\[
\Gamma = \sigma m \sum_{s=0}^{\omega-1} b(s), \quad \chi = M \sum_{s=0}^{\omega-1} b(s), C(r) = \max \{f(x) : x \in \mathbb{R}_+, \|x\| \leq r\} > 0.
\]

**Lemma 2.5.** Assume that (H1), (H2) hold. For any \(\eta > 0\) and \(x \in K\{0\}\), if there exists \(f\) such that \(f(x(k)) \geq x(k)\eta\) for \(k \in [0, \omega]\), then \(\|Tx\| \geq \lambda \Gamma \eta \|x\|\).
Proof. Since \( x \in K \setminus \{0\} \) and \( f(x(k)) \geq x(k)\eta \) for \( k \in [0, \omega] \), we have

\[
Tx(k) = \lambda \sum_{s=k}^{k+\omega-1} G(k, s)b(s)f(x(s - \tau(s))) \\
\geq \lambda m \sum_{s=0}^{\omega-1} b(s)f(x(s - \tau(s))) \\
\geq \lambda m \sum_{s=0}^{\omega-1} b(s)x(k)\eta \\
\geq \lambda m \sum_{s=0}^{\omega-1} b(s)\sigma \|x\| \eta
\]

Thus \( \|Tx\| \geq \lambda \Gamma \eta \|x\| \). This completes the proof.

Let \( \hat{f} : [1, \infty) \to \mathbb{R}_+ \) be the function given by

\[
\hat{f}(\theta) = \max \{ f(x) : x \in \mathbb{R}_+, and 1 \leq \|x\| \leq \theta \}.
\]

It is easy to see that \( \hat{f}(\theta) \) is nondecreasing function on \( [1, \infty) \). The following lemma is essentially the same as Lemma 3.6 in [19] and Lemma 2.8 in [20].

**Lemma 2.6.** ([19, 20]) Assume (H2) holds. If \( \lim_{x \to \infty} \frac{f(x)}{x} \) exists (which can be infinity) then \( \lim_{\theta \to \infty} \frac{\hat{f}(\theta)}{\theta} \) exists and \( \lim_{\theta \to \infty} \frac{\hat{f}(\theta)}{\theta} = \lim_{x \to \infty} \frac{f(x)}{x} \).

**Lemma 2.7.** Assume that (H1) and (H2) hold. Let \( r > \frac{1}{\sigma} \) and if there exists an \( \varepsilon > 0 \) such that \( \hat{f}(r) \leq \varepsilon r \), then \( \|Tx\| \leq \lambda \chi \varepsilon \|x\| \) for \( x \in \partial \Omega_r \).

Proof. From the definition of \( T \) for \( x \in \partial \Omega_r \), we have

\[
\|Tx\| \leq \lambda M \sum_{s=0}^{\omega-1} b(s)f(x(s - \tau(s))) \\
\leq \lambda M \sum_{s=0}^{\omega-1} b(s)\hat{f}(r)\eta \\
\leq \lambda M \sum_{s=0}^{\omega-1} b(s)\varepsilon r \\
\leq \lambda M \sum_{s=0}^{\omega-1} b(s)\varepsilon \|x\|.
\]

Thus \( \|Tx\| \leq \lambda \chi \varepsilon \|x\| \).
In view of definition $C(r)$, it follows that $0 < f(x(k)) \leq C(r)$ for $k \in [0, \omega]$, if $x \in \partial \Omega_r, r > 0$. Thus it is easy to see the following lemma can be shown in similar manner as in Lemma 2.7.

**Lemma 2.8.** Assume (H1), (H2) hold. If $x \in \partial \Omega_r, r > 0$ then $\|Tx\| \leq \lambda \chi C(r)$.

**Proof.** From the definition of $T$ for $x \in \partial \Omega$ we have

\[
\|Tx\| \leq \lambda M \sum_{s=0}^{\omega-1} b(s) f(x(s - \tau(s))) \\
\leq \lambda M \sum_{s=0}^{\omega-1} b(s) C(r) \\
\leq \lambda \chi C(r).
\]

This implies $\|Tx\| \leq \lambda \chi C(r)$.

\[\square\]

3 Main Results

In this section, we establish conditions for the existence and multiplicity of positive periodic solutions of (1).

**Theorem 3.1.** Let (H1), (H2) hold. We assume that $\lim_{x \to 0} f(x) = \infty$.

(a) If $\lim_{x \to \infty} \frac{f(x)}{x} = 0$ then for all $\lambda > 0$ (1) has a positive solution.

(b) If $\lim_{x \to \infty} \frac{f(x)}{x} = \infty$, then for all small $\lambda > 0$ (1) has two positive solutions.

(c) If there exists a $\lambda_0 > 0$ such that (1) has a positive periodic solution for $0 < \lambda < \lambda_0$.

**Proof.** (a). From the assumptions, $\lim_{x \to 0} f(x) = \infty$ there is an $r_1 > 0$ such that

\[ f(x) \geq \eta x \]

for $x \in K \setminus \{0\}$ and $0 < x < r_1$, where $\eta > 0$ is chosen so that

\[ \lambda \Gamma \eta > 1. \]

Let $\Omega_{r_1} = \{ x \in K : \|x\| < r_1 \}$. If $x \in \partial \Omega_{r_1}$, then

\[ f(x(k)) \geq x(k) \eta. \]
Lemma 2.5 implies that
\[ \|Tx\| \geq \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for } x \in \partial \Omega_{r_1}. \] (4)

We now determine \( \Omega_{r_2} \). Let \( \Omega_{r_2} = \{ x \in K : \|x\| < r_2 \} \). Note that \( \lim_{x \to -\infty} \frac{f(x)}{x} = 0 \), it follows from Lemma 2.6, \( \lim_{\theta \to -\infty} \frac{f(\theta)}{\theta} = 0 \). Therefore there is an \( r_2 > \max \{2r_1, \frac{1}{\sigma}\} \) such that
\[ \hat{f}(r_2) \leq \varepsilon r_2, \]
where the constant \( \varepsilon > 0 \) satisfies
\[ \lambda \varepsilon \chi < 1. \]
Thus, we have by Lemma 2.7 that
\[ \|Tx\| \leq \lambda \varepsilon \|x\| < \|x\| \quad \text{for } x \in \partial \Omega_{r_2}. \] (5)

By Lemma 2.1 applied to (4) and (5), it follows that \( T \) has a fixed point in \( \bar{\Omega}_{r_2} \setminus \Omega_{r_1} \), which is the desired positive solution of (1). \( \square \)

**Proof.** (b). Fix two numbers \( 0 < r_3 < r_4 \), there exists a \( \lambda_0 \) such that
\[ \lambda_0 < \frac{r_3}{\chi C(r_3)}, \quad \lambda_0 < \frac{r_4}{\chi C(r_4)}, \]
where \( \chi C(r) \) defined in Lemma 2.8. Thus, in Lemma 2.8 implies that, for \( 0 < \lambda < \lambda_0 \),
\[ \|Tx\| \leq \lambda \chi C(r_j) \leq \frac{r_j}{\chi C(r_j)} \chi C(r_j) = r_j = \|x\|. \]
Thus
\[ \|Tx\| < \|x\| \quad \text{for } x \in \partial \Omega_{r_j}, \quad (j = 3, 4). \] (6)

On the other hand, in view of the assumptions \( \lim_{x \to -\infty} \frac{f(x)}{x} = \infty \) and \( \lim_{x \to 0} f(x) = \infty \), there are positive numbers \( 0 < r_2 < r_3 < r_4 < \hat{H} \) such that
\[ f(x) \geq \eta x \]
for \( x \in K \setminus \{0\} \) and \( 0 < x \leq r_2 \) or \( \|x\| \geq \hat{H} \) where \( \eta > 0 \) is chosen so that
\[ \lambda \Gamma \eta > 1. \]
Thus if \( x \in \partial \Omega_{r_2} \), then
\[ f(x) \geq \eta x. \]
Let \( r_1 = \max \left\{ 2r_4, \frac{\bar{H}}{\sigma} \right\} \) if \( x \in \partial \Omega_{r_1} \), then
\[
\min_{k \in [0, \omega]} x(k) \geq \sigma \|x\| = \sigma r_1 = \bar{H},
\]
which implies that
\[
f(x) \geq \eta x.
\]
Lemma 2.5 implies that
\[
\|Tx\| \geq \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for} \quad x \in \partial \Omega_{r_1},
\]
and
\[
\|Tx\| \geq \lambda \Gamma \eta \|x\| > \|x\| \quad \text{for} \quad x \in \partial \Omega_{r_2}.
\]
It follows from Lemma 2.1 applied to (6), (7) and (8), \( T \) has two fixed points \( x_1 \) and \( x_2 \) such that \( x_1 \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2} \) and \( x_2 \in \bar{\Omega}_{r_1} \setminus \Omega_{r_4} \), which are the desired distinct positive periodic solutions of (1) for \( \lambda < \lambda_0 \) satisfying
\[
r_2 < \|x_1\| < r_3 < r_4 < \|x_2\| < r_1.
\]

**Proof.** (c). Choose a number \( r_3 > 0 \). By Lemma 2.8 we infer that there exists a \( \lambda_0 = \frac{r_3}{\chi_{C(r_3)}} > 0 \) such that
\[
\|Tx\| < \|x\| \quad \text{for} \quad x \in \partial \Omega_{r_3} \quad 0 < \lambda < \lambda_0.
\]
On the other hand, in view of assumption \( \lim_{x \to 0} f(x) = \infty \), there exists a positive number \( 0 < r_2 < r_3 \) such that
\[
f(x) \geq \eta x
\]
for \( x \in K \setminus \{0\} \) and \( 0 < x < r_2 \) where \( \eta > 0 \) is chosen so that
\[
\lambda \Gamma \eta > 1.
\]
Thus if \( x \in \partial \Omega_{r_2} \), then
\[
f(x) \geq \eta x.
\]
Lemma 2.5 implies that
\[
\|Tx\| \geq \lambda \Gamma \eta \|x\| > \|x\|, \quad \text{for} \quad x \in \partial \Omega_{r_2}.
\]
It follows from Lemma 2.1 applied to (9) and (10), that \( T \) has a fixed point \( x \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2} \). The fixed point \( x \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2} \) is the desired positive periodic solution of (1).

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