Galerkin Method for the Numerical Solution of Hypersingular Integral Equations Based Chebyshev Polynomials

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\textbf{Abstract}

This paper is concerned with finding approximate solution for the hypersingular integral equations. It is well known that hypersingular integrals are exist, if the density function $f$ satisfied Hölder-continuous first derivative, i.e $f \in C^{1,\alpha}[-1,1]$. Relating the hypersingular integrals to Cauchy principal-value integrals, we expand the kernel and the density function of hypersingular integral equation by the sum of Chebyshev polynomial of the second kind. The orthogonal property take apart in the simplification of this approach. The exactness of the approximate method is proved. Numerical examples show the accuracy and effectiveness of the present work. Comparison with other researchers is presented.

\textbf{Keywords:} Hypersingular integral equation, Singular integrals, Chebyshev polynomials, Galerkin method

\section{Introduction}

Chebyshev polynomials are encountered in virtually every area of numerical analysis, and they hold particular importance in many subjects such as orthogonal polynomials, polynomial approximation, numerical integration, and
special methods [1]. There are several kinds of Chebyshev polynomials, in this paper we involve with the first and second kind polynomials \(T_n(x), U_n(x)\)

\[
T_n(x) = \cos n\theta, \quad x = \cos \theta, \quad -1 \leq x \leq 1, \quad n = 0, 1, 2, \cdots,
\]

\[
U_{n-1}(x) = \frac{\sin n\theta}{\sin \theta}, \quad x = \cos \theta, \quad -1 \leq x \leq 1, \quad n = 0, 1, 2, \cdots.
\]

The close relationship of Chebyshev polynomials with the trigonometric functions 'sine' and 'cosine' gives the strong power for this kind of the polynomials.

The dual boundary element method is based on singular and hypersingular integral equations where the integral equations involve improper integrals, in which the integrands are unbounded and could belong to a class of functions where the Riemann integration is not possible. Thus the result of the integration simply means the finite-part of the divergent integral [2]. A general hypersingular integral equation of the first kind, over a finite interval, can be represented by

\[
\int_{-1}^{1} \phi(t) \left[ \frac{K(t, x)}{(t-x)^2} + L(t, x) \right] dt = f(x), \quad x \in (-1, 1),
\]

where \(K(t, x)\) and \(L(t, x)\) are regular square-integrable functions of \(t\) and \(x\), and \(K(x, x) \neq 0\). The density function \(\phi(t)\) satisfies the Hölder-continuous first derivative, \(\phi(t) \in C^{1,\alpha}[-1, 1]\) means that there are constants \(|A| < \infty\) and \(0 < \alpha \leq 1\), such that the following inequality holds [2]

\[
|\phi(t) - \phi(x) - \phi'(t)(t-x)| \leq A|t-x|^\alpha+1.
\]

The hypersingular integral in Eq. (1) is defined as Hadamard finite-part integral of the function \(g(t) \in C^{1,\alpha}[-1, 1]\), [3]

\[
\int_{a}^{b} \frac{g(t)}{(x-t)^2} dt = \lim_{\varepsilon \to 0} \left[ \int_{a}^{x-\varepsilon} + \int_{x+\varepsilon}^{b} \right] \frac{g(t)}{(x-t)^2} dt - \frac{2g(x)}{\varepsilon},
\]

where, \(g(t) = \phi(t)K(t, x)\) in which the neighbourhood \(\varepsilon\) is symmetric about the singular point.

Many researchers concentrate on dominant hypersingular integral equations (HSIEs) of the following type [4]

\[
\int_{-1}^{1} \frac{g(t)}{(t-x)^2} dt = f(x), \quad x \in (-1, 1),
\]

It is known that, the differentiation of Cauchy principal-value integral (CPVI) results in hypersingular integrals (HSI) [5],

\[
\int_{-1}^{1} \frac{g(t)}{(t-x)^2} dt = \frac{d}{dx} \int_{-1}^{1} \frac{g(t)}{(t-x)} dt, \quad x \in (-1, 1),
\]
this relation explained shortly that the HSI represents a natural extension of CPVI, which defined as \[3\]
\[
\int_{-1}^{1} \frac{\varphi(t)}{x-t} \, dt = \lim_{\varepsilon \to 0} \left( \int_{-1}^{x-\varepsilon} \frac{\varphi(t)}{x-t} \, dt + \int_{x+\varepsilon}^{1} \frac{\varphi(t)}{x-t} \, dt \right).
\]

Hence, in the evaluation of finite-part integrals, we use either integration by parts or the method of singularity subtraction [6].

Integral equation methods are known for solving many problems of fracture mechanics, acoustics, elasticity and fluid mechanics, because of their ability to resolve the strong singularities arise in stress fields close to the endpoints on the smooth outer boundaries or interfaces where the boundary conditions change type [7]. In recent years, numerous works have been focusing on the development of more advanced and efficient methods for integral equations such as implicity linear collocation methods [8, 9, 10] and semianalytical-numerical techniques such as Adomian’s decomposition method [11].

In [10], Ioakimidis used the classical collocation and Galerkin methods for the numerical solution of Fredholm integral equations of the second kind with a double pole singularity of the form
\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}g(t)}{(x-t)^2} \, dt + \int_{-1}^{1} \sqrt{1-t^2}k(t,x)g(t) \, dt = -f(x), \quad -1 < x < 1,
\]
where the density function \(g(t)\) is proportional to the crack opening displacement function along the crack. Ioakimidis was interested in the values of the stress intensity factors \(K(t)\) at the crack tips, where
\[
K(\pm 1) = g(\pm 1),
\]

taking into account that
\[
U_n(\pm 1) = (\pm 1)^n(n + 1),
\]

implies
\[
K(\pm 1) = \sum_{i=0}^{n} (\pm 1)^i(i + 1)a_i.
\]

Eq. (7) determines both the numerical values for stress intensity factors at the crack tips \(K(\pm 1)\) [12, 13], and also the unknown coefficients \(a_i\), where
\[
K(1) = 1.83122498, \quad K(-1) = 0.70090677,
\]

Mandal and Bhattacharya [14] obtained approximate solutions for Fredholm integral equations of second kind and hypersingular kernels using Bernstein polynomials as basis.
Our aim is to obtain higher order accuracy of expansion method for hypersingular integral equation in Hilbert space, by combining both of Chebyshev truncated series and the hypersingular kernel expansion methods. Where the expansion of density function \( g(t) \) in Eq.(2) based on the second kind of Chebyshev polynomial \( U_i(t) \), and the HSI kernel expansion is based on the singular integral kernel expansion using the relation in (3). However, we have used the Galerkin approximation, which has several important advantages over the more widely used collocation method \([15, 16]\). This approach is not only competitive with collocation, but can be faster for sufficiently large-scale calculations. It has the ability to evaluate simply, without ambiguity, hypersingular intervals \([17]\) using standard continuous elements. With collocation, the existence of these integrals depends upon a numerically difficult constraint. Other benefits of Galerkin include higher accuracy, simple and reliable analysis at corners, and a well developed mathematical theory.

The rest of this paper is organized as follows: Section 2, the details of approximate methods are given. The exactness of the proposed method is proved in Section 3, while in Section 4, several examples are given to show the accuracy of the developed technique.

## 2 Description of the numerical method

Consider the characteristic hypersingular kernel equation

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{g(t)dt}{(t-x)^2} = f(x), \quad |x| < 1, \quad (9)
\]

Let the unknown function \( g(t) \) in (9), which is bounded at both endpoints \( x = \pm 1 \), be written as

\[
g(t) = \sqrt{1-t^2} \varphi(t), \quad (10)
\]

where \( \varphi(t) \) is a well defined function of \( t \), approximated by a finite sum of Chebyshev polynomial \( U_i(t) \) as

\[
\varphi(t) \approx \sum_{i=0}^{n} C_i U_i(t), \quad (11)
\]

where \( C_i, \ i = 0, 1, 2, \cdots, n, \) are the unknown coefficients to be determined.

The singular integral kernel can be easily approximated by \([15]\)

\[
\frac{1}{t-x} \approx \sum_{j=0}^{m} k_j(x)U_j(t), \quad (12)
\]
where the unknown function $k_j(x)$ can easily be found by applying the following property

$$\int_{-1}^{1} \frac{\sqrt{1 - t^2} U_n(t)}{x - t} \, dt = \pi T_{n+1}(x),$$

(13)

along with the orthogonal condition

$$\int_{-1}^{1} \sqrt{1 - t^2} U_i(t) U_j(t) \, dt = \frac{\pi}{2} \delta_{ij},$$

(14)

where $\delta_{ij}$ is the kronecker delta, which yields

$$K_i(x) = -2T_{i+1}(x).$$

(15)

Substituting (15) into (12) and differentiate with respect to $x$, yields

$$\frac{1}{(t - x)^2} \approx -2 \sum_{j=0}^{m} (j + 1) U_j(x) U_j(t),$$

(16)

Due to (11) and (16) Eq. (9) is reduced to the following algebraic equation

$$- \sum_{i=0}^{n} (i + 1) C_i U_i(x) = f(x).$$

(17)

(i.) For Galerkin method, the unknown coefficients $C_i$ in Eq. (17) are defined by multiplying (17) by $\sqrt{1 - t^2} U_j(t)$, then integrate both sides over [-1,1], with the use of (14), so that

$$C_i = \frac{-2}{\pi (i + 1)} \int_{-1}^{1} \sqrt{1 - x^2} f(x) U_j(x) \, dx,$$

(18)

(ii.) For the collocation method, we have the following system of equations

$$- \sum_{i=0}^{n} (i + 1) C_i U_i(x_j) = f(x_j), \quad j = 0, 1, \cdots, n,$$

(19)

choosing the roots of $T_{n+1}(x)$ as the collocation points $x_j$ along the interval [-1,1], which are

$$x_j = \cos \left( \frac{2k + 1}{2(n + 1)} \pi \right), \quad k = 0, 1, \cdots, n.$$
we obtain a system of \( n+1 \) linear equations with the \( n+1 \) unknown coefficients \( C_i \). The calculation of \( C_i \) permits the evaluation of \( g_n(t) \) in (10). Approximate solution of (9) is

\[
g_n(x) = \sqrt{1 - x^2} \sum_{i=0}^{n} C_i U_i(x),
\]

where \( C_i \) are calculated by (19) for the collocation method and (18) for Galerkin method.

### 3 Exactness of the numerical solution

In this section, the exactness of the numerical solution in Eq. (21) is shown as follow:

**Lemma 3.1.** Let \( f \) in the Eq. (9) be a linear function \( f(x) = ax + b \), then the numerical solution in (21) is exact.

**Proof:**

Let the function \( f \) in (9) be a linear, i.e.

\[
f(x) = ax + b, \quad -1 < x < 1,
\]

where \( a, b \) are known constants. Then Eq. (9) becomes

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{g(t)dt}{(t-x)^2} = ax + b, \quad |x| < 1.
\]

According to [4, 14], the exact solution of (9) is

\[
g(x) = \frac{1}{\pi^2} \int_{-1}^{1} f(t) \ln \left| \frac{x-t}{1-xt-\sqrt{(1-x^2)(1-t^2)}} \right| dt, \quad -1 \leq x \leq 1.
\]

For the special case when \( f(x) = ax + b \), the exact solution of (9) is found to be

\[
g(x) = \frac{-1}{2} \sqrt{1 - x^2} (2b + ax)
\]

The unknown coefficients of (21) can be obtained by substituting (22) into (18),

\[
C_i = \frac{-2}{\pi(i+1)} \int_{-1}^{1} \sqrt{1 - x^2} (ax + b)U_j(x)dx,
\]
by applying the orthogonal relation in (14), we found that
\[ C_0 = -b, \quad C_1 = -\frac{a}{4}, \quad C_i = 0, \quad i \geq 2. \] (26)
Substituting (26) into (21) for \( n = 2 \), we have
\[ g_2(x) = \sqrt{1 - x^2} (C_0 U_0(x) + C_1 U_1(x)), \]
it is easy to see that
\[ g_2(x) = -\frac{1}{2} \sqrt{1 - x^2} (2b + ax), \]
which is identical to the exact solution •

Lemma 3.2. Let \( f \) in the Eq. (9) be a polynomial of degree 3, \( f(x) = ax^3 + bx^2 + cx + d \), then the numerical solution in (21) is exact.

Proof: Let the function \( f \) in (9) be a linear, i.e.
\[ f(x) = ax^3 + bx^2 + cx + d, \quad -1 < x < 1, \]
where \( a, b, c, d \) are known constants. Then Eq. (9) becomes
\[ \frac{1}{\pi} \int_{-1}^{1} g(t) \frac{dt}{(t-x)^2} = ax^3 + bx^2 + cx + d, \quad |x| < 1. \]
According to [4, 14], the exact solution of (9) is
\[ g(x) = -\sqrt{1 - x^2} \left( \frac{a}{4} x^3 + \frac{b}{3} x^2 + \frac{a + 4c}{8} x + \frac{b}{6} + d \right) \] (27)
Due to (14) and (18), we found that
\[ C_0 = -\left(\frac{b}{4} + d\right), \quad C_1 = -\left(\frac{a}{8} + \frac{c}{4}\right), \quad C_2 = -\frac{b}{12}, \quad C_3 = -\frac{a}{32}, \quad C_i = 0, \quad i \geq 4. \] (28)
Substituting (28) into (21) for \( n = 3 \), we have
\[ g_3(x) = \sqrt{1 - x^2} \sum_{i=0}^{3} C_i U_i(x). \] (29)
It is not difficult to verify that, the numerical solution (29) is identical to the exact solution in (27) which complete the proof •
4 Illustrative examples

In this section, we illustrate the above method to obtain numerical solution of HSIE.

Example 1. Let us consider the following HSIE

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{g(t)dt}{(t-x)^2} = -(3x^2 + \frac{1}{2}), \quad |x| < 1, \quad (30)
\]

where the known function \( f(x) \) in Eq. (9) is quadratic, and with simple substitution in the relation (3), we can get the exact solution of (30)

\[
g(x) = \sqrt{1-x^2}(1 + x^2). \quad (31)
\]

The bounded approximate solution of Eq. (30) defined in (21) depends on Chebyshev polynomial of the second kind (1), which gives,

\[
x = \frac{1}{2} U_1(x), \quad x^2 = \frac{1}{4}(U_2(x) + U_0(x)), \quad x^3 = \frac{1}{8}(U_3(x) + 2U_1(x)), \ldots .
\]

With the use of (14) and (18), yields

\[
C_0 = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2}(3x^2 + \frac{1}{2})U_0(x)dx
\]

\[
= \frac{6}{\pi} \int_{-1}^{1} \sqrt{1-x^2} \left[ \frac{1}{4}(U_2(x) + U_0(x)) \right] U_0(x)dx + \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-x^2} dx = \frac{5}{4}
\]

and by the same procedure, we get

\[
C_1 = 0, \quad C_2 = \frac{1}{4}, \quad C_i = 0, i \geq 3. \quad (32)
\]

Substitute the above results into (21), we obtain the numerical solution of Eq. (30) which is the same as the exact solution in (31).

Example 2.

For the special case when \( f(x) \) in (3) is a polynomial of degree 3,

\[
f(x) = 8x^3 - 2x + 1, \quad (33)
\]

the exact solution of (9) is found to be

\[
g(x) = -\sqrt{1-x^2}(1 + 2x^3). \quad (34)
\]
The unknown coefficients of (21) can be obtained after substituting (33) into (18),

\[ C_i = \frac{-2}{\pi (i + 1)} \int_{-1}^{1} \sqrt{1 - x^2} (8x^3 - 2x + 1) U_j(x) dx, \]

by considering the orthogonal relation in (14), we found that

\[ C_0 = -1, \quad C_1 = -\frac{1}{2}, \quad C_2 = 0, \quad C_3 = -\frac{1}{4}, \quad C_i = 0, \quad i \geq 4 \quad (35) \]

Substituting (35) into (21) for \( n = 3 \), yields Eq. (34).

**Example 3.**

Consider \( f(x) = -e^x \)

\[ \frac{1}{\pi} \int_{-1}^{1} g(t) dt (t - x)^2 = -e^x, \quad |x| < 1, \quad (36) \]

where \( e^x \) expanded by Taylor-polynomial

\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \frac{1}{5040} x^7 + \frac{1}{40320} x^8. \quad (37) \]

This problem has already considered in [10], here we will use the hypersingular integral equation (2) with the determination of (8) at the crack tips \( t = \pm 1 \). For Galerkin method the coefficients \( C_i \) are calculated by (18).

<table>
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<th>( n )</th>
<th>( K(1) ) Galerkin method (21)</th>
<th>( K(1) ) Galerkin method [10]</th>
<th>( K(-1) ) Galerkin method (21)</th>
<th>( K(-1) ) Galerkin method [10]</th>
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Theoretical value \( K(1) = 1.83122498 \) \( K(-1) = 0.70090677 \)
The results presented in Table 1, shows a very good convergence of the values of the stress intensity factor to the theoretical values. Whereas, the convergence of the result by this method gives 8 corrected decimal digits for $n = 7$. Where the values of $K(\pm 1)$ are simply calculated from (21) after solving the linear system in (18) for $C_i$. Maple 12 is used for the computations of the results.

5 Conclusion

A simple method of approximating the density function and the hypersingular kernel, using the finite series of Chebyshev polynomials is presented, for solving characteristic hypersingular integral equations. We reformulated the main problem in (2), as a set of linear algebraic equations that solved by applying the usual Galerkin method. The numerical tests indicate that this method provided 8 corrected decimal digits for $n = 7$. Combined with the advantages that Galerkin method presents in the simple solution of the linear system of equations, this technique provides faster and efficient approximation algorithm by increasing $n$.

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References


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