

Theorem of Coincidence of Classes for a Generalised Shift Operator

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Abstract. In this paper, for a generalised shift operator introduced earlier, we prove theorem of coincidence of classes of functions defined by the order of best approximation by algebraical polynomials and the generalised Lipschitz classes defined by the generalised shift operator.

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1. INTRODUCTION

In [4, 5], a generalised shift operator was introduced, by its means the generalised moduli of smoothness of given order were defined, and Jackson's type theorem was proved for these moduli.

In the present paper, we prove the theorem of coincidence of classes of functions defined by the order of best approximation by algebraic polynomials and the generalised Lipschitz classes defined by means of the generalised moduli of smoothness.

2. DEFINITIONS

Denote by L_p , $1 \leq p < \infty$, the set of functions f measurable in sense of Lebesgue with summable p -th power, by L_∞ the set of functions f bounded almost everywhere in $[-1, 1]$, and

$$\|f\|_p = \begin{cases} \left(\int_{-1}^1 |f(x)|^p dx \right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \text{ess sup}_{-1 \leq x \leq 1} |f(x)|, & \text{for } p = \infty. \end{cases}$$

Denote by $L_{p,\alpha}$ the set of functions f such that $f(x)(1-x^2)^\alpha \in L_p$, and put

$$\|f\|_{p,\alpha} = \|f(x)(1-x^2)^\alpha\|_p.$$

Denote by $E_n(f)_{p,\alpha}$ the best approximation of a function $f \in L_{p,\alpha}$ by algebraic polynomials of degree not greater than $n-1$, in $L_{p,\alpha}$ metrics, i.e.,

$$E_n(f)_{p,\alpha} = \inf_{P_n} \|f - P_n\|_{p,\alpha},$$

where P_n is an algebraic polynomial of degree not greater than $n-1$.

By $E(p, \alpha, \lambda)$ we denote the class of functions $f \in L_{p,\alpha}$ satisfying the condition

$$E_n(f)_{p,\alpha} \leq Cn^{-\lambda},$$

where $\lambda > 0$ and C is a constant not depending on n ($n \in \mathbb{N}$).

Define generalised shift operator $\hat{\tau}_t(f, x)$ by

$$\hat{\tau}_t(f, x) = \frac{1}{\pi(1-x^2)\cos^4 \frac{t}{2}} \int_0^\pi B_{\cos t}(x, \cos \varphi, R) f(R) d\varphi,$$

where

$$(2.1) \quad \begin{aligned} R &= x \cos t - \sqrt{1-x^2} \sin t \cos \varphi, \\ B_y(x, z, R) &= 2 \left(\sqrt{1-x^2}y + xz\sqrt{1-y^2} \right. \\ &\quad \left. + \sqrt{1-x^2}(1-y)(1-z^2) \right)^2 - (1-R^2). \end{aligned}$$

By means of this generalised shift operator define the r -th generalised difference by

$$\begin{aligned} \Delta_t^1(f, x) &= \Delta_t(f, x) = \hat{\tau}_t(f, x) - f(x), \\ \Delta_{t_1, \dots, t_r}^r(f, x) &= \Delta_{t_r}(\Delta_{t_1, \dots, t_{r-1}}^{r-1}(f, x), x) \quad (r = 2, 3, \dots), \end{aligned}$$

and for a function $f \in L_{p,\alpha}$, define the r -th generalised modulus of smoothness as follows

$$\hat{\omega}_r(f, \delta)_{p,\alpha} = \sup_{\substack{|t_j| \leq \delta \\ j=1,2,\dots,r}} \|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p,\alpha} \quad (r = 1, 2, \dots).$$

Consider the class $H(p, \alpha, r, \lambda)$ of functions $f \in L_{p,\alpha}$ satisfying the condition

$$\hat{\omega}_r(f, \delta)_{p,\alpha} \leq C\delta^\lambda,$$

where $\lambda > 0$ and C is a constant not depending on δ .

Put $y = \cos t$, $z = \cos \varphi$ in the operator $\hat{\tau}_t(f, x)$, denote it by $\tau_y(f, x)$ and rewrite it in the form

$$\tau_y(f, x) = \frac{4}{\pi(1-x^2)(1+y)^2} \int_{-1}^1 B_y(x, z, R) f(R) \frac{dz}{\sqrt{1-z^2}},$$

where R and $B_y(x, z, R)$ are defined in (2.1)

Define the r -th power of the generalised shift operator by

$$\begin{aligned}\tau_y^1(f, x) &= \tau_y(f, x), \\ \tau_{y_1, \dots, y_r}^r(f, x) &= \tau_{y_r}(\tau_{t_1, \dots, t_{r-1}}^{r-1}(f, x), x) \quad (r = 2, 3, \dots).\end{aligned}$$

By $P_\nu^{(\alpha, \beta)}(x)$ ($\nu = 0, 1, \dots$) we denote the Jacobi polynomials, i.e., the algebraic polynomials of degree ν , orthogonal with the weight function $(1-x)^\alpha(1+x)^\beta$ on the segment $[-1, 1]$, and normed by the condition

$$P_\nu^{(\alpha, \beta)}(1) = 1 \quad (\nu = 0, 1, \dots).$$

Denote by $a_n(f)$ the Fourier–Jacobi coefficients of a function f , integrable with the weight function $(1-x^2)^2$ on the segment $[-1, 1]$, with respect to the system of Jacobi polynomials $\left\{P_n^{(2,2)}(x)\right\}_{n=0}^\infty$, i.e.,

$$a_n(f) = \int_{-1}^1 f(x) P_n^{(2,2)}(x) (1-x^2)^2 dx \quad (n = 0, 1, \dots).$$

3. AUXILIARY STATEMENTS

In order to prove our results we need the following theorem.

Theorem 3.1. *Let the numbers p and α be such that $1 \leq p \leq \infty$;*

$$\begin{aligned}1/2 < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < 3/2 & \quad \text{for } p = \infty.\end{aligned}$$

If $f \in L_{p,\alpha}$, then for every natural number n

$$C_1 E_n(f)_{p,\alpha} \leq \hat{\omega}_r(f, 1/n)_{p,\alpha},$$

where the positive constant C_1 does not depend on f and n .

Theorem 3.1 was proved in [5]. It is known as a Jackson's type theorem.

We also need the following lemmas.

Lemma 3.1. *The operator $\tau_y(f, x)$ has the following properties:*

- 1) *it is linear,*
- 2) $\tau_1(f, x) = f(x),$
- 3) $\tau_y\left(P_\nu^{(2,2)}, x\right) = P_\nu^{(2,2)}(x) P_\nu^{(0,4)}(y) \quad (\nu = 0, 1, \dots),$
- 4) $\tau_y(1, x) = 1,$
- 5) $a_n(\tau_y(f, x)) = a_n(f) P_n^{(0,4)}(y) \quad (n = 0, 1, \dots).$

Lemma 3.1 was proved in [4]

Lemma 3.2. *Let the numbers p and α be such that $1 \leq p \leq \infty$;*

$$\begin{aligned} 1/2 < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < 3/2 & \quad \text{for } p = \infty. \end{aligned}$$

If $f \in L_{p,\alpha}$, then

$$\|\hat{\tau}_t(f, x)\|_{p,\alpha} \leq \frac{C}{\cos^4 \frac{t}{2}} \|f\|_{p,\alpha},$$

where constant C does not depend on f and t .

Lemma 3.2 was proved in [4].

Corollary 3.1. *Let be given numbers p , α and r such that $1 \leq p \leq \infty$, $r \in \mathbb{N}$;*

$$\begin{aligned} \frac{1}{2} < \alpha \leq 1 & \quad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \quad \text{for } 1 < p < \infty, \\ 1 \leq \alpha < \frac{3}{2} & \quad \text{for } p = \infty. \end{aligned}$$

Let $f \in L_{p,\alpha}$. The following inequality holds true

$$\|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p,\alpha} \leq \frac{C}{\prod_{j=1}^r \left(\cos \frac{t_j}{2}\right)^4} \|f\|_{p,\alpha},$$

where constant C does not depend on f and t_j ($j = 1, 2, \dots, r$).

Proof. By applying induction with respect to r , it is not difficult to see (for the analogous property of the ordinary difference for $y_1 = y_2 = \dots = y_r$ see, e.g., [6, f. 102]) that the operator $\Delta_{t_1, \dots, t_r}^r(f, x)$ can be written in the following form

$$\Delta_{t_1, \dots, t_r}^r(f, x) = \sum_{k=1}^r (-1)^{k-1} \sum_{i_1 < \dots < i_k} \tau_{\cos y_{i_1}, \dots, \cos y_{i_k}}^k(f, x) + (-1)^r f(x),$$

i.e. as a linear combination of powers $\hat{\tau}_{t_{i_1}, \dots, t_{i_k}}^k(f, x)$ ($i_1 < i_2 < \dots < i_k$; $k = 0, 1, \dots, r$) of the appropriate generalised shift operator. Now, Corollary 3.1 is proved by applying r times Lemma 3.2. \square

4. STATEMENT OF RESULTS

Theorem 4.1. *Let be given numbers p , α , r and λ such that $1 \leq p \leq \infty$, $r \in \mathbb{N}$;*

$$1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty,$$

$$1 \leq \alpha < \frac{3}{2} \quad \text{for } p = \infty$$

and $0 < \lambda < 2r$. Let $f \in L_{p,\alpha}$. If

$$E_n(f)_{p,\alpha} \leq Mn^{-\lambda},$$

then

$$\hat{\omega}_r(f, \delta)_{p,\alpha} \leq CM\delta^\lambda,$$

where constant C does not depend on f , M and δ .

Proof. Let $P_n(x)$ be an algebraical polynomial of degree not greater than $n-1$ such that

$$\|f - P_n\|_{p,\alpha} = E_n(f)_{p,\alpha} \quad (n = 1, 2, \dots).$$

We define algebraical polynomials $Q_k(x)$ by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \dots)$$

and $Q_0(x) = P_1(x)$. Since for $k \geq 1$

$$\begin{aligned} \|Q_k\|_{p,\alpha} &= \|P_{2^k} - P_{2^{k-1}}\|_{p,\alpha} \leq \|P_{2^k} - f\|_{p,\alpha} + \|f - P_{2^{k-1}}\|_{p,\alpha} \\ &= E_{2^k}(f)_{p,\alpha} + E_{2^{k-1}}(f)_{p,\alpha}, \end{aligned}$$

then by the conditions of the theorem we have

$$(4.1) \quad \|Q_k\|_{p,\alpha} \leq C_1 M 2^{-k\lambda}.$$

Taking into consideration property 4) in Lemma 3.1 of the operator τ_y , without lost of generality we may suppose that $t_s \neq 0$ ($s = 1, 2, \dots, r$). For $0 < |t_s| \leq \delta$ ($s = 1, 2, \dots, r$) we estimate

$$I = \|\Delta_{t_1, \dots, t_r}^r(f, x)\|_{p,\alpha}.$$

For every positive integer N , taking into account property 1) in Lemma 3.1 and the fact that linearity of the operator $\tau_t(f, x)$ implies linearity of $\tau_{t_1, \dots, t_r}^r(f, x)$, and, in turn, linearity of $\Delta_{t_1, \dots, t_r}^r(f, x)$; we get

$$I \leq \|\Delta_{t_1, \dots, t_r}^r(f - P_{2^N}, x)\|_{p,\alpha} + \|\Delta_{t_1, \dots, t_r}^r(P_{2^N}, x)\|_{p,\alpha}.$$

Since

$$P_{2^N}(x) = \sum_{k=0}^N Q_k(x),$$

we have

$$\begin{aligned} I &\leq \|\Delta_{t_1, \dots, t_r}^r (f - P_{2^N}, x)\|_{p, \alpha} + \sum_{k=0}^N \|\Delta_{t_1, \dots, t_r}^r (Q_k, x)\|_{p, \alpha} \\ &= J + \sum_{k=1}^N I_k. \end{aligned}$$

Let N be chosen in such a way that

$$(4.2) \quad \frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}.$$

We prove the following inequalities

$$(4.3) \quad J \leq C_2 M \delta^\lambda$$

and

$$(4.4) \quad I_k \leq C_3 M 2^{-k\lambda},$$

where constants C_2 and C_3 do not depend on f , M , δ and k .

First we consider J . By Corollary 3.1, taking into account that $|t_1| \leq \delta$, we have

$$\begin{aligned} \|\Delta_{t_1, \dots, t_r}^r (f - P_{2^N}, x)\|_{p, \alpha} &\leq \frac{C_4}{\prod_{j=1}^r \left(\cos \frac{t_j}{2}\right)^4} \|f - P_{2^N}\|_{p, \alpha} \\ &= C_5 E_{2^N}(f)_{p, \alpha} \end{aligned}$$

Therefrom, the condition of the theorem and inequality (4.2) yield

$$\|\Delta_{t_1, \dots, t_r}^r (f - P_{2^N}, x)\|_{p, \alpha} \leq C_6 M 2^{-N\lambda} \leq C_7 M \delta^\lambda,$$

which proves inequality (4.3).

Now we prove inequality (4.4). Note that, taking into consideration Corollary 3.1, we have

$$\|\Delta_{t_1, \dots, t_r}^r (Q_k, x)\|_{p, \alpha} \leq \frac{C_8}{\prod_{j=1}^r \left(\cos \frac{t_j}{2}\right)^4} \|Q_k\|_{p, \alpha}.$$

Hence,

$$I_k \leq \frac{C_9}{\prod_{j=1}^r \left(\cos \frac{t_j}{2}\right)^4} M 2^{-k\lambda},$$

which proves inequality (4.4).

Inequalities (4.3), (4.4) and (4.2) yield

$$I \leq C_{10} M \left(\delta^\lambda + \sum_{k=1}^N 2^{-k\lambda} \right) \leq C_{11} M (\delta^\lambda + 2^{-N\lambda}) \leq C_{12} M \delta^\lambda.$$

Theorem 4.1 is proved. \square

Theorem 4.2. *Let be given numbers p , α , r and λ such that $1 \leq p \leq \infty$, $\lambda > 0$, $r \in \mathbb{N}$;*

$$1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty,$$

$$1 \leq \alpha < \frac{3}{2} \quad \text{for } p = \infty$$

Let $f \in L_{p,\alpha}$. If

$$\hat{\omega}_r(f, \delta)_{p,\alpha} \leq M\delta^\lambda,$$

then

$$E_n(f)_{p,\alpha} \leq CMn^{-\lambda},$$

where constant C does not depend on f , M and n .

Proof. Let $\delta = \frac{1}{n}$. Then, taking into account Theorem 3.1, we obtain

$$E_n(f)_{p,\alpha} \leq \frac{1}{C_1} \hat{\omega}_r \left(f, \frac{1}{n} \right)_{p,\alpha} \leq CMn^{-\lambda}.$$

Theorem 4.2 is proved. \square

Theorem 4.3. *Let be given numbers p , α , r and λ such that $1 \leq p \leq \infty$, $r \in \mathbb{N}$;*

$$1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty,$$

$$1 \leq \alpha < \frac{3}{2} \quad \text{for } p = \infty$$

Then for $0 < \lambda < 2r$ the classes of functions $H(p, \alpha, r, \lambda)$ coincide between themselves for different values of r , and they coincide with the class $E(p, \alpha, \lambda)$.

Proof. Note that, under the condition of the theorem, Theorem 4.2 implies the inclusion

$$H(p, \alpha, r, \lambda) \subseteq E(p, \alpha, \lambda),$$

while Theorem 4.1 implies the converse inclusion

$$E(p, \alpha, \lambda) \subseteq H(p, \alpha, r, \lambda).$$

Hence we conclude that the assertion of Theorem 4.3 is implied by Theorems 4.2 and 4.1. \square

Note that analogues of Theorems 4.2, 4.1 and 4.3 for another generalised shift operator were proved in [1] and, in more general forms, in [3, 2].

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