Average of Some Multiplicative Functions

on the Set of Integers without Large Prime Factor

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Abstract

Let $\lambda > 0$, $\delta \in ]0, 1[$ and $f(n)$ be a multiplicative function satisfying essentially
for a large $x$; $\sum_{f(p) \leq x} \log f(p) = \lambda x + O(x(\log x)^{-\delta})$. In the present work, we estab-
lish an asymptotic formula for the two sums $\sum_{f(n) \leq x; P(n) \leq y} \frac{1}{f(n)}$ and $\sum_{f(n) \leq x; P(n) \leq y} 1$
, valid in the domain $1 \leq cx \leq C_0(\log f(y))^{-\delta}$, for a suitable constants $c$ and $C_0$.

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1 Introduction

Let $P(n)$ denote the largest prime divisor of a positive integer $n$, with $P(1) = 1$. Also, throughout the paper, the letters $p$, $p_1$ and $p_2$ denote a prime number.

For $x \geq 1$ and $y > 1$, we introduce the quantity:

$$F_f(x, y) = \sum_{f(n) \leq x, P(n) \leq y} \frac{1}{f(n)}, \quad S_f(x, y) = \sum_{f(n) \leq x, P(n) \leq y} 1, \quad \psi_f(x, y) = \sum_{n \leq x, P(n) \leq y} f(n).$$

Obviously, $S_f(x, y)$, $F_f(x, y)$, and $\psi_f(x, y)$ are a natural generalization of the well-known function, usually denoted by $\psi(x, y)$, which is the number of integers $\leq x$ and free of prime factors $> y$. The later has been the object of a number of articles in the last three decades (see for example: [2], [1], [5], [10], [4] et [7]).

In [6], Naimi gives an asymptotic formula for $S_f(x, y)$, for a multiplicative function such that $f(p) = \frac{p}{\lambda}$, $(\lambda > 1)$:

$$S_f(x, y) = F.x. (\log(\frac{y}{\lambda}))^{\lambda - 1} \rho_\lambda(v) \{1 + O_\varepsilon(\frac{\log_2 x + \log(v + 1)}{\log(\frac{y}{\lambda})})\}$$

uniformly in the domain $H_{\varepsilon, \lambda, c}(x_0)$, defined by

$$x \geq x_0, \quad \exp(\log_2 cx)^{5/3 + \varepsilon} \leq \frac{y}{\lambda} \leq cx$$

with $\varepsilon > 0$, $c = \frac{1}{\inf\{f(n); n \geq 1\}}$, $v = \frac{\log cx}{\log y/\lambda}$ and $\varphi_\lambda$ to be the continuous solution of the differential difference equation:

$$u\varphi'_\lambda(u) = (\lambda - 1)\varphi_\lambda(u) - \lambda\varphi_\lambda(u - 1), \quad u > 1, \quad \lambda > 0,$$

with initial conditions

$$\varphi_\lambda(u) = \begin{cases} 
0 & \text{si } u \leq 0 \\
\frac{u^{\lambda - 1}}{\Gamma(\lambda)} & \text{si } 0 < u \leq 1
\end{cases}$$

The function $\varphi_\lambda$, generalizes Dickman's function $\varphi$. Many paper have been written on proprieties of this function (see for example [8] et [3]).
In his paper [9], J.M. Song established an asymptotic formula for \( \psi_f(x, y) \) where \( f \) is non-negative multiplicative arithmetic function satisfying:

There exists \( \delta \in ]0, 1[ \), \( \lambda > 0 \) and \( b > 0 \) such as

\[
\sum_{p \leq z} \frac{f(p)}{p} \log p = \lambda \log z + O((\log z)^{1-\delta}), \quad z \geq 2
\]

and

\[
\sum_{k \geq 2} \frac{f(p^k)}{p^k} \log p^k \leq b.
\]

The aim of this paper is to give estimates of the two sums \( F_f(x, y) \) and \( S_f(x, y) \) for a class of arithmetic function where the principals conditions are inspired from [9].

Let \( \xi_{\lambda, \delta} \) denote the class of non-negative multiplicative function \( f \) satisfying the following conditions:

\[
\begin{align*}
(\Omega_1) & \quad \sum_{f(p) \leq z} \log f(p) = \lambda z + O(z(\log z)^{-\delta}); \quad \lambda \geq 1, \quad \delta \in ]0, 1[ \\
(\Omega_2) & \quad \sum_{f(p^k) \leq z} \frac{\log f(p^k)}{f(p^k)} \ll (\log z)^{-\delta} \\
(\Omega_3) & \quad p_1 < p_2 \Rightarrow p_1 \ll f(p_1) < f(p_2)
\end{align*}
\]

Hypothesis \( (\Omega_1) \) is often replaced by the weaker hypothesis

\[
(\Omega_1^*) \quad \sum_{f(p) \leq z} \frac{\log(f(p))}{f(p)} = \lambda \log z + O((\log z)^{1-\delta}),
\]

obtained from \( (\Omega_1) \) using Abel summation.

Before starting our results, we introduce some notations.

For \( f \in \xi_{\lambda, \delta} \), we note \( v = \frac{\log cx}{\log f(y)} \) (in this article, we confuse \( y \) and the largest prime number less than \( y \)), \( c = \frac{1}{\inf \{f(n), n \geq 1\}} \) (\( c \geq 1 \)) and throughout the paper, \( O \) and \( \ll \) depends at most than \( \lambda \).

Let \( j_\lambda \) the continuous solution of the difference-differential equation

\[
u j_\lambda'(u) = kj_\lambda(u) - \lambda j_\lambda(u - 1), \quad u > 1, \quad \lambda > 0,
\]
with initial conditions:

\[ j_\lambda(u) = \begin{cases} 0 & \text{si } u \leq 0 \\ \frac{u^\lambda}{\Gamma(\lambda+1)} & \text{si } 0 < u \leq 1 \end{cases} \]

The function \( j_\lambda \), verifying \( j_\lambda'(v) = \rho_\lambda(v) \), was studied in [9]. We give its principal properties in Lemma 1.

Our main result is the following:

**Theorem 1:**

Let \( f \in \xi_{\lambda, \delta} \). There exist a constant \( A(f) \) such as for all sufficiently large \( x \):

\[
S_f(x, y) = A(f)(\log f(y))^{\lambda-1} \rho_\lambda(v) \{ 1 + O\left( \frac{\log v}{\inf(\log f(y), \log x)}^\delta \right) \}
\]

for a suitable constant \( C_0 \).

The following theorem is the generalization of (1):

**Theorem 2:**

Let \( f \in \xi_{\lambda, \delta} \). For all sufficiently large \( x \)

\[
S_f(x, y) = A(f)x(\log f(y))^{\lambda-1} \rho_\lambda(v) \{ 1 + O\left( \frac{\log v}{\inf(\log f(y), \log x)}^\delta \right) \}
\]

uniformly in \((H'_{f, \delta, C_0})\), where \( A(f) \) is the constant of theorem 1.

2 Preliminary lemmas

**Lemma 1 :** Properties of \( j_\lambda \):
1) \( j_k(u) \) is a strictly increasing function in \([0, +\infty[\) \hspace{1cm} (2.1.1)

2) \( j_k(u) \) converges to 1 as \( u \to +\infty \) \hspace{1cm} (2.1.2)

3) \( j_\lambda(u) \geq \frac{1}{2} \), for \( u > \lambda \) \hspace{1cm} (2.1.3)

4) For \( v \geq 0 \) and \( a \in ]0, v[ \):

\[
\frac{j_\lambda(v)}{v^\lambda} - \frac{j_\lambda(a)}{a^\lambda} = -\lambda \int_a^v \frac{j_\lambda(x-1)}{x^{\lambda+1}} dx
\]  \hspace{1cm} (2.1.4)

5) For \( v \in ]1, 2[ \):

\[
j_\lambda(v) = \frac{1}{\Gamma(\lambda + 1)} [v^\lambda - \lambda \int_1^v \frac{(v-h)^\lambda}{h} dh]
\]  \hspace{1cm} (2.1.5)

6) For \( v \geq 1 \):

\[
j_\lambda(v) = v^\lambda [\frac{1}{\Gamma(\lambda + 1)} - \lambda \int_1^v \frac{j_\lambda(u-1)}{u^{\lambda+1}} du]
\]  \hspace{1cm} (2.1.6)

7) For \( v \geq 1 \):

\[
j_\lambda(v) = \frac{\lambda}{v} \int_{v-1}^v j_\lambda(t) dt + \frac{1}{v} \int_1^v j_\lambda(t) dt + \frac{1}{v(\lambda + 1) \Gamma(\lambda + 1)}.
\]  \hspace{1cm} (2.1.7)

Proof:

1), 2), 3) and 7) :see [11].

4) We have \((x^{-\lambda} j_\lambda(x))' = \frac{1}{x^{\lambda+1}} [-\lambda j_\lambda(x) + x j_\lambda'(x)] = \frac{1}{x^{\lambda+1}} (-\lambda j_\lambda(x - 1))\).

When we integrate the two sides of this equality with respect to \( x \) from \( a \) to \( v \), we obtain the result.

5) The proof is deduced from 4) in the particular case \( v \in ]1, 2[ \) and \( a = 1 \). Indeed

\[
\frac{j_\lambda(v)}{v^\lambda} - \frac{1}{\Gamma(\lambda + 1)} = -\frac{\lambda}{\Gamma(\lambda + 1)} \int_1^v \frac{(x-1)^\lambda}{x^{\lambda+1}} dx
\]

\[\implies j_k(v) = \frac{1}{\Gamma(\lambda + 1)} [v^\lambda - \lambda \int_1^v \frac{1}{x(v - x)^\lambda} dx].\]

With the change of variable \( h = \frac{x}{z} \) we obtain:

\[
j_\lambda(v) = \frac{1}{\Gamma(\lambda + 1)} [v^\lambda - \lambda \int_1^v \frac{(v-h)^\lambda}{h} dh], \quad (v \in ]1, 2[).\]
This proves (2.1.5)
6) Taking \( a = 1 \) in the equation (2.1.4).

**Lemma 2:**

Let \( f \in \xi_{\lambda, \delta} \). We have

1)

\[
L(y, t) = \lambda \log_2 t - \lambda \log_2(f(y)) + O((\log f(y))^{-\delta}), \hspace{1em} (t > 0)
\]  
(2.2.1)

2)

\[
\sum_{f(p) \leq x} \frac{\log f(p)}{(f(p))^2} = \int_{1/c}^{x} \frac{1}{t} d \left( \sum_{f(p) \leq t} \frac{\log f(p)}{f(p)} \right) \ll (\log x)^{1-\delta}
\]  
(2.2.2)

3)

\[
\sum_{f(p^k) \leq x \atop k \geq 2} \frac{1}{f(p^k)} \ll (\log x)^{-\delta}.
\]  
(2.2.3)

**Proof:**

1) by Abel summation based on \((\Omega_1^*)\) we deduced that

\[
L(y, t) = \sum_{f(p) \leq t} \frac{1}{f(p)} = \int_{f(y)}^{t} \frac{1}{\log x} d \left( \sum_{f(p) \leq x} \frac{\log f(p)}{f(p)} \right)
\]

\[= \lambda \log_2 t - \lambda \log_2(f(y)) + O((\log f(y))^{-\delta}).\]

2) Using Abel summation based on \((\Omega_1^*)\), we obtain (2.2.2)

3) We obtain (2.2.3) by \((\Omega_2)\).

**Lemma 3:**

Suppose \( f \in \xi_{\lambda, \delta} \). Then for \( v \in [1, \lambda + 2] \) we have

\[
\sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)}.(\log \frac{x}{f(p)})^\lambda = \lambda.(\log f(y))^\lambda.\left[ \int_{1}^{v} \frac{(v-h)^\lambda}{h} dh + O(\frac{1}{(\log f(y))^{\delta}}) \right].
\]  
(2.3)

**Proof:**
Let $M$ denote the sum on the left hand side of the lemma. By Abel summation, we get:

$$M = \int_{f(y)}^{cx} (\log\left(\frac{x}{t}\right))^\lambda \cdot d\left(\sum_{f(y) < f(p) \leq t} \frac{1}{f(p)}\right) = \int_{f(y)}^{cx} (\log(\frac{x}{t}))^\lambda \cdot d(L(y, t))$$

$$= \left[(\log(\frac{x}{t}))^\lambda \cdot L(y, t)\right]_{t=f(y)}^{t=cx} + \lambda \int_{f(y)}^{cx} \frac{t}{t} \cdot \frac{(\log(\frac{x}{t}))^{\lambda-1}}{t} L(y, t) \, dt = A + B$$

$$\Rightarrow M = A + B.$$  

The quantity $A$ is equal to

$$(\log(\frac{x}{cx}))^\lambda \cdot [\lambda \log_2 cx - \lambda \log_2 f(y) + O((\log x)^{-\delta})] - (\log(\frac{x}{f(y)}))^\lambda \cdot [\lambda \log_2 f(y) - \lambda \log_2 f(y)] + O((\log x)^{-\delta})]$.$

And $B$ is equal to

$$-[\log(\frac{x}{t}))^\lambda \cdot \log_2(t)]_{t=f(y)}^{t=cx} + \lambda \int_{f(y)}^{cx} \frac{(\log(\frac{x}{t}))^\lambda}{t \log t} \, dt$$

$$+ \lambda \log_2 f(y)[(\log(\frac{x}{t}))^\lambda]_{t=f(y)}^{t=cx} + O((\log(f(y)))^{\lambda-\delta}).$$

After simplification, we get

$$M = \int_{f(y)}^{cx} \frac{(\log(\frac{x}{t}))^\lambda}{t \log t} \, dt + O((\log f(y))^{\lambda-\delta}).$$

Let $G$ denote the integral

$$\int_{f(y)}^{cx} \frac{(\log(\frac{x}{t}))^\lambda}{t \log t} \, dt$$

$$G = \int_{f(y)}^{cx} \frac{(\log(\frac{x}{t}))^\lambda}{t \log t} \, dt$$

$$= \lambda (\log f(y))^\lambda [1 + O(\frac{1}{\log f(y)})] \cdot \int_{f(y)}^{cx} (v - \frac{\log ct}{\log f(y)})^\lambda \cdot \frac{1}{t \log ct} \, dt.$$  

With the change of variable $h = \frac{\log ct}{\log f(y)}$, we obtain

$$G = \lambda (\log f(y))^\lambda [1 + O(\frac{1}{\log f(y)})] \cdot \int_{1+\frac{\log ct}{\log f(y)}}^{\frac{\log ct}{\log f(y)}} (v - h)^\lambda \cdot \frac{1}{h} \, dh.$$
Let
\[ U = \int_{v}^{v+\frac{\log c_{\lambda,\delta}}{\log f(y)}} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh. \]

We remarks that
\[ U = \int_{1}^{v} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh - \int_{1}^{v+\frac{\log c_{\lambda,\delta}}{\log f(y)}} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh \]
\[ = \int_{1}^{v} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh - \int_{1}^{v+\frac{\log c_{\lambda,\delta}}{\log f(y)}} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh - U_{1} - U_{2}. \]

It’s clearly that \( U_{1} \) and \( U_{2} \) are at most \( \ll \frac{1}{\log f(y)} \), then
\[ G = \lambda (\log f(y))^{\lambda} [1 + O\left(\frac{1}{\log f(y)} \right)] \left[ \int_{1}^{v} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh + O\left(\frac{1}{\log f(y)} \right) \right] \]
\[ = \lambda (\log f(y))^{\lambda} \left[ \int_{1}^{v} (v - h)^{\lambda} \frac{h}{\log f(y)} \, dh + O\left(\frac{1}{\log f(y)} \right) \right]. \]

This proves the lemma

**Lemma 4:**

Suppose \( f \in \xi_{\lambda,\delta} \) and \( v \in [0, \lambda + 2] \), then
\[ F_{f}(x, y) = F_{f}(x) - \sum_{f(y) < f(p) \leq \epsilon x} \frac{1}{f(p)} F_{f}\left(\frac{x}{f(p)}\right) p + O((\inf(\log f(y), \log x))^{\lambda-\delta}) \quad (2.4) \]

where \( F_{f}(x) = F_{f}(x, x) = \sum_{f(n) \leq x} \frac{1}{f(n)}. \)

Proof

\[ F_{f}(x, y) = F_{f}(x) - \sum_{f(n) \leq x} \frac{1}{f(n)} = F_{f}(x) - S_{0}. \]

When we rearrange the sum \( S_{0} \) following the largest prime divisor of \( n \), we obtain
\[ S_{0} = \sum_{p \geq y} \sum_{f(n) \leq x \atop P(n) = p} \frac{1}{f(n)}. \]
Average of some multiplicative functions

\begin{align*}
&= \sum_{p > y} \sum_{f(n) \leq x} \frac{1}{f(n)} + \sum_{p > y} \sum_{f(n) \leq x; P(n) = p} \frac{1}{f(n)} \\
&= \sum_{p > y} \frac{1}{f(p)} \sum_{f(m) \leq \frac{x}{p}} \frac{1}{f(m)} + \sum_{p > y} \sum_{f(p^k) \leq x; k \geq 2} \frac{1}{f(p)} \sum_{f(m) \leq \frac{x}{f(p^k)}} \frac{1}{f(m)} \\
&= \sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)} \cdot f_f \left( \frac{x}{f(p)}, p \right) - \sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)} \sum_{f(m) \leq \frac{x}{f(p^k)}} \frac{1}{f(m)} = T_p + R_1 + R_2.
\end{align*}

We verify that \( R_1 \ll \frac{F_f(cx)}{f(y)} \ll (\log x)^{\lambda - \delta} \). And by Abel summation, \( R_2 \) is at most

\[
\int_{f(y)}^{cx} \frac{(\log \frac{x}{t})^\lambda}{\log t} dt \left( \sum_{f(p^k) \leq t; k \geq 2} \frac{\log f(p^k)}{f(p^k)} \right) \ll (\log x)^{\lambda - \delta}.
\]

This proves (2.4)

**Lemma 5:**

Let \( f \in \xi_{\lambda, \delta} \). For \( v \in [1, \lambda + 2] \)

\[
\sum_{f(y) < f(p) \leq cx} \frac{(\log f(p))^\lambda j_\lambda(v_p - 1)}{f(p)} = \lambda v^\lambda (\log f(y))^\lambda \int_1^v \frac{j_\lambda(u - 1)}{u^{\lambda + 1}} du + O((\log f(y))^{\lambda - \delta});
\]

 où \( v_p = \frac{\log cx}{\log f(p)} \).

**Proof:** Let \( I \) denote the sum on the left side of (2.5) and a double integration by parts applied to \( I \) gives

\[
I = \int_{f(y)}^{cx} (\log t)^\lambda \cdot j_\lambda \left( \frac{\log cx}{\log t} - 1 \right) dt \left( \sum_{f(y) < f(p) \leq t} \frac{1}{f(p)} \right) = \lambda \int_{f(y)}^{cx} j_\lambda \left( \frac{\log cx}{\log t} - 1 \right)(\log t)^\lambda \frac{dt}{t \log t} + O((\log f(y))^{\lambda - \delta}).
\]
With the change of variable \( u = \frac{\log cx}{\log t} \), we obtain (2.5).

**Lemma 6:**

\[
(\log cx)F_f(x, y) \leq (\log c).F_f(x, y) + \int_{\frac{1}{c}}^{x} \frac{F_f(t, y)}{t} dt + \sum_{\frac{1}{c} \leq f(p^k) \leq cx \atop p \leq y : k \geq 1} F_f(\frac{x}{f(p^k)}, y) \frac{\log f(p^k)}{f(p^k)};
\]

\((v \geq 1, f \in \xi)\).

**Proof:**

This inequality can be easily established by evaluating the sum

\[
S = \sum_{f(n) \leq x \atop P(n) \leq y} \frac{\log(\log f(n))}{f(n)};
\]

in two different ways: On the one hand, we have:

\[
S = \int_{\frac{1}{c}}^{x} \log ct \, d(F_f(t, y)) = (\log cx).F_f(x, y) - \int_{\frac{1}{c}}^{x} \frac{F_f(t, y)}{t} dt \quad (*).
\]

On the other hand, partial summation yields

\[
S = (\log c).F_f(x, y) + \sum_{f(n) \leq x \atop P(n) \leq y} \frac{1}{f(n)} \sum_{p \leq y} \frac{\log f(p^k)}{f(p^k)} \sum_{\frac{1}{c} \leq f(p) \leq cx : k \geq 1} \frac{1}{f(m)}
\]

\[
\Rightarrow S = (\log c).F_f(x, y) + \sum_{p \leq y} \frac{\log f(p^k)}{f(p^k)} \sum_{\frac{1}{c} \leq f(m) \leq f(p^k) : k \geq 1} \frac{1}{f(m)} \sum_{P(m) \leq y : (m, p) = 1} \frac{1}{f(m)}
\]

\[
\Rightarrow S \leq (\log c).F_f(x, y) + \sum_{p \leq y} \frac{\log f(p^k)}{f(p^k)} \sum_{1 \leq f(p^k) \leq cx : k \geq 1} F_f(\frac{x}{f(p^k)}, y) \quad (**)
\]

and the result follows on equating the two expressions (*) and (**).

**Lemma 7:**

Suppose \( f \in \xi \) and for \( \theta \in ]0, 1[ \); we have:

\[
\sum_{1 \leq f(p) \leq (f(y))^\theta} j_\lambda(v - \frac{\log f(p)}{\log f(y)}) \frac{\log f(p)}{f(p)} = \lambda \log f(y) \cdot \int_0^\theta j_\lambda(v - u) du + O( (\log f(y))^{1-\delta} ).
\]

(2.7)
Proof:

We consider \( s(z) = \sum_{f(p) \leq z} \frac{\log f(p)}{f(p)} \) and \( r(z) = s(z) - \lambda \log z \).

One deduces from \((\Omega^*_1)\) that there exists a constant \( A > 0 \) such that:
\[
|r(z)| \leq A \cdot (\log z)^{1-\delta}.
\]

Denote the left-hand side of the above formula by \( S \). Partial summation yields
\[
S = \int_{1}^{(f(y))^\theta} j_\lambda(v - \frac{\log t}{\log f(y)}) \, d(s(t))
\]
\[
= \lambda \int_{1}^{(f(y))^\theta} j_\lambda(v - \frac{\log t}{\log f(y)}) \frac{dt}{t} + \int_{1}^{(f(y))^\theta} j_\lambda(v - \frac{\log t}{\log f(y)}) \, ds(t) - \lambda \log t
\]
\[
= M + R.
\]

With the change of variable : \( u = \frac{\log t}{\log f(y)} \), we obtain
\[
M = \lambda \log f(y) \cdot \int_{0}^{\theta} j_\lambda(v - u) \, du.
\]

The term \( R \) is at most of order
\[
j_\lambda(v - \theta). |r((f(y))^\theta)| - \int_{1}^{(f(y))^\theta} r(t) \frac{d}{dt}[j_\lambda(v - \frac{\log t}{\log f(y)})] \, dt
\]
\[
\quad \Rightarrow \ll j_\lambda(v - \theta). |r((f(y))^\theta)| + (\log f(y))^{1-\delta} \int_{1}^{(f(y))^\theta} j'_\lambda(v - \frac{\log t}{\log f(y)}) \, dt
\]
\[
\quad \ll j_\lambda(v - \theta). |r((f(y))^\theta)| + (\log f(y))^{1-\delta}(j_\lambda(v - \theta) - j_\lambda(v))
\]
\[
\quad \ll (\log f(y))^{1-\delta}.
\]

This complete the proof of Lemma.

3 Proof of theorem 1

Proposition 0 :

Let \( f \in \xi_{\lambda,\delta} \). Then for all sufficiently large \( x \):
\[
F_f(x) = C(f) \cdot (\log x)^\lambda + O((\log x)^{\lambda-\delta}); \quad \lambda \geq 1, \quad \delta \in (0, 1]
\]  
(3.1)
where \( C(f) = \frac{A(f)}{\Gamma(\lambda + 1)} \) and \( A(f) \) defined in theorem 1.

**Proof of proposition 0:**

On the one hand, we have

\[
\sum_{f(n) \leq x} \frac{\log f(n)}{f(n)} = \sum_{f(n) \leq x} \frac{1}{f(n)} \sum_{p^k \mid n} \log f(p^k) = \sum_{f(m)f(p^k) \leq x} \frac{\log f(p^k)}{f(p^k)f(m)} = D. \tag{P.1}
\]

On the other hand, partial summation yield

\[
\sum_{f(n) \leq x} \frac{\log f(n)}{f(n)} = \int_{1/c}^{x} \log t \, d(F_f(t)) = (\log x)F_f(x) - \int_{1/c}^{x} F_f(t) \frac{dt}{t}. \tag{P.2}
\]

The sum on the right of (P.1) is equal to

\[
\sum_{f(m)f(p^k) \leq x} \frac{\log f(p^k)}{f(m)f(p^k)} = D_1 + D_2
\]

after separating terms corresponding to \( k = 1 \) and \( k \geq 2 \). By \((\Omega_2)\), the sum \( D_2 \) is at most

\[
\sum_{f(m) \leq x} \frac{1}{f(m)} \sum_{p,k \geq 2} \frac{\log(f(p^k))}{f(p^k)} \ll F_f(x).
\]

The quantity \( D_1 \) is equal to

\[
\sum_{f(m) \leq x} \frac{1}{f(m)} \sum_{f(p) \leq x/\gamma(m)} \frac{\log f(p)}{f(p)} - \sum_{f(l)f(p^k) \leq x} \frac{\log f(p)}{f(l)f(p)f(p^k)} = B_1 + B_2
\]

Define

\[
T(x) = \int_{1/c}^{x} \frac{F_f(t)}{t} dt = \sum_{f(m) \leq x} \frac{\log(x/\gamma(m))}{f(m)}. \tag{P.3}
\]

So that, by \((\Omega_1^\dagger)\)

\[
B_1 = \sum_{f(m) \leq x} \frac{1}{f(m)} \{ \lambda \log(x/\gamma(m)) + O((\log x)^{1-\delta}) \} = \lambda T(x) + O(F_f(x)(\log x)^{1-\delta}).
\]
Using (2.2.2) and (2.2.3), the sum $B_2$ is at most
\[
\sum_{f(l)(f(p))^2 \leq x} \frac{\log f(p)}{f(l)(f(p))^2} + \sum_{f(l)(f(p))(f(p^k)) \leq x} \frac{\log f(p)}{f(l)f(p)(f(p^k))} \lesssim \sum_{f(l) \leq x} \frac{1}{f(l)} \sum_{f(p) \leq \sqrt{x}} \frac{\log f(p)}{(f(p))^2} + \sum_{f(l) \leq x} \frac{1}{f(l)} \sum_{(f(p))(f(p^k)) \leq cx} \frac{\log f(p)}{f(p)(f(p^k))} \lesssim F_f(x)(\log x)^{1-\delta}.
\]

Finally, after appeal to (P.1), (P.2) and (P.3), we conclude that
\[
(\log x)F_f(x) = T(x) + O(F_f(x)) + [\lambda T(x) + O(F_f(x)(\log x)^{1-\delta})].
\]

And, for
\[
\varepsilon(x) \ll (\log x)^{-\delta} \quad (P.4)
\]

we have
\[
F_f(x) = \frac{\lambda + 1}{\log x} T(x) + F_f(x)\varepsilon(x).
\]

If $x$ is a large enough, say $x \geq x_0$, then
\[
|\varepsilon(x)| \leq 1/2, \quad x \geq x_0 \quad (P.5)
\]

and we have the useful expression
\[
F_f(x) = \frac{1}{1-\varepsilon(x)} \frac{\lambda + 1}{\log x} T(x), \quad x \geq x_0. \quad (P.6)
\]

Let
\[
E(x) = \log\left(\frac{(\lambda + 1)T(x)}{(\log x)^{\lambda+1}}\right)
\]

and note that, by (P.3) and (P.6)
\[
E'(x) = \frac{(\lambda + 1)\varepsilon(x)}{x \log x(1-\varepsilon(x))} \ll \frac{1}{(\log x)^{1+\delta}}, \quad x \geq x_0.
\]

Hence $E_0 = \int_{1+}^{\infty} E'(x)dx$ converge absolutely, and therefore
\[
\int_x^{\infty} E'(z)dz = \int_{1+}^{\infty} E'(z)dz - \int_{1+}^{x} E'(z)dz = E_0 - (E(x) - E(1^+)).
\]
On writing \( C(f) = \exp(E_0 + E(1^+)) \)
\[
\exp(E(x)) = \frac{(\lambda + 1).T(x)}{(\log x)^{(\lambda + 1)}} = C(f). \exp(- \int_x^\infty E'(z)dz) = C(f)[1 + O((\log x)^{-\delta})].
\]
When we substitute this in (P.6) and use (P.4), we obtain
\[
F_f(x) = C(f). (\log x)^{\lambda} + O((\log x)^{\lambda-\delta}); \quad \lambda \geq 1, \quad \delta \in ]0, 1[. \quad (P.7)
\]
To determine \( C(f) \), we agree as follows. For \( s > 1 \) and \( u \in \mathbb{R} \), by Abel summation, we have
\[
\sum_{f(n) \leq u} \frac{1}{(f(n))^s} = \int_{1/c}^u \frac{1-s}{c} d(F_f(t)) = F_f(u)u^{1-s} + (s-1). \int_{1/c}^u \frac{F_f(x)}{x^s}dx. \quad (P.8)
\]
Using (P.7)
\[
F_f(u)u^{1-s} = u^{1-s}[C(f). (\log u)^{\lambda} + O((\log u)^{\lambda-\delta})]
\]
then
\[
\lim_{u \to \infty} F_f(u)u^{1-s} = 0. \quad (P.9)
\]
The quantity on the right of (P.8) is equal (by (P.7)) to
\[
(s-1)[\int_1^{+\infty} \frac{C(f)(\log x)^{\lambda}}{x^s}dx + O(\int_{1/c}^{+\infty} \frac{C(f)(\log x)^{\lambda-\delta}}{x^s}dx+1)] = C(f).\Gamma(\lambda+1)(s-1)^{-\lambda}+O(s-1).
\]
When we substitute this in (P.8) and using (P.9), we obtain
\[
\sum_{f(n) \geq 1/c} \frac{1}{f(n)^s} = C(f). \frac{\Gamma(\lambda+1)}{(s-1)^\lambda} + O(s-1).
\]
So that
\[
C(f) = \frac{1}{\Gamma(\lambda+1)}. \lim_{s \to 1^+} [(s-1)^\lambda. \sum_{f(n) \geq 1/c} \frac{1}{f(n)^s}].
\]
On the other hand \( \lim_{s \to 1^+} [(s-1)\zeta(s)] = 1 \), hence
\[
C(f) = \frac{1}{\Gamma(\lambda+1)}. \lim_{s \to 1^+} [(\prod_p (1-p^{-s})^{-\lambda}).( \sum_{f(n) \geq 1/c} \frac{1}{f(n)^s})].
\]
This proves the proposition.
Theorem 1 is obtained by combining the two following propositions.

**Proposition 1:**
Let \( f \in \xi_{\lambda, \delta} \). For \( v \in ]0, \lambda + 2[ \), we have
\[
F_f(x, y) = A(f)(\log(f(y))^\lambda j_{\lambda,v}(v)\{1 + O((\inf(\log f(y),\log x)^{-\delta})\}).
\] (3.2)

**Proposition 2:**
Let \( f \in \xi_{\lambda, \delta} \). For \( \lambda + 2 \leq v \leq \exp(\frac{1}{\lambda_0}\log(f(y))) \)
\[
F_f(x, y) = A(f).\log(f(y))^\lambda j_{\lambda,v}(v).[1 + O(\frac{\log v}{\log(f(y))^\delta})].
\] (3.3)

**Proof of Proposition 1:**
We proceed inductively, we will proves that
\[
\text{For } y \geq y_0, \text{ for all } k \in [1, \lambda + 2] \text{ and for all } v \in ]k-1, k[, \text{ we have :}
\]
\[
(R) \quad F_f(x, y) = A(f)(\log(f(y))^\lambda j_{\lambda,v}(v)\{1 + O((\inf(\log f(y),\log x)^{-\delta})\}).
\]

Let \( T_p \) denote the sum on the right hand side of (2.4).

- \( k = 1, v \in ]0, 1[ \iff 1 < cx \leq f(y) \).

We verify that \( T_p = 0 \), hence the expression (2.4) became :
\[
F_f(x, y) = F_f(x) + O((\log x)^{\lambda-\delta}) = F_f(x) \quad (P.1.1).
\]

And we remarks that \((\frac{\log x}{\log cx})^\lambda = 1 + O(\frac{1}{\log x})\), then we deduce from (3.1) that
\[
F_f(x, y) = A(f)\frac{v^\lambda}{\Gamma(\lambda + 1)}(\log f(y))^\lambda[1 + O((\log x)^{-\delta})] \quad (y \geq y_0) \quad (P.1.2)
\]

which establishes the equality (R) for \( k = 1 \), since \( j_k(v) = \frac{v^\lambda}{\Gamma(\lambda + 1)} \text{ in } ]0, 1[ \).

- \( k = 2, v \in ]1, 2[ \iff f(y) \leq cx \leq (f(y))^2 \).

In this case, we remarks that for \( f(y) < f(p) \), the \( F_f(\frac{x}{f(p)}, p) \) verify (P.1.1), hence
\[
T_p = C(f) \sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)}(\log(f(y))^\lambda + O(\sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)}(\log(f(y))^\lambda-\delta))) = M + R.
\]

Using (2.2.1), the error term bellows is at most
\[
\sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)}(\log(f(y))^\lambda-\delta) \ll (\log f(y))^\lambda-\delta \quad \sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)} \ll (\log f(y))^\lambda-\delta.
\]
and after appeal to lemma 3, the sum on (P.1.2) is equal to
\[ C(f) \lambda (\log f(y))^\lambda \left[ \int_1^v \frac{(v-h)^\lambda}{h} dh + O\left( \frac{1}{(\log f(y))^\delta} \right) \right]. \]
Hence
\[ T_p = C(f)M + R = C(f) \lambda (\log f(y))^\lambda \left[ \int_1^v \frac{(v-h)^\lambda}{h} dh + O\left( \frac{1}{(\log f(y))^\delta} \right) \right]. \]
When we substitute the expression of \( T_p \) in (2.4), we have
\[ F_f(x, y) = F_f(x) - C(f) \lambda (\log f(y))^\lambda \int_1^v \frac{(v-h)^\lambda}{h} dh + O((\log(f(y))^{\lambda - \delta}) \]
and using (2.1.5) and proposition 0, we obtain:
\[ F_f(x, y) = A(f)(\log f(y))^\lambda j_\lambda(v)[1 + O((\log f(y))^{-\delta})]. \]
This proves (R) for \( k = 2 \). Let \( k \in [2, \lambda + 1] \); assume that (R) is verified for all integers \( k_0 \in [1, k] \) and we have to prove that (R) is also true for \( k + 1 \). For this, let \( v \in [k, k + 1] \).

We remarks that for \( f(p) > f(y) \), \( v_p = \frac{\log f(y)}{\log f(p)} = \frac{\log cx}{\log f(p)} - 1 \leq v - 1 \leq k \), hence, when we applying the induction hypothesis to the term \( F_f(\frac{x}{f(p)}, p) \), we obtain
\[ F_f(\frac{x}{f(p)}, p) = A(f)(\log f(p))^\lambda j_\lambda(v_p)[1 + O((\log f(y))^{-\delta})]. \]
So
\[ T_p = A(f)[ \sum_{f(y) < f(p) \leq cx} \frac{\log f(p)^\lambda j_\lambda(v_p)}{f(p)} + O\left( \sum_{f(y) < f(p) \leq cx} \frac{\log f(p)^\lambda j_\lambda(v_p)}{f(p)}(\log f(y))^{-\delta} \right) ] \]
\[ = A(f)[I + R]. \]
On the one hand, using lemma 5, we have
\[ I = \lambda v^\lambda (\log f(y))^\lambda \int_1^v j_\lambda\left( \frac{u-1}{u^{\lambda+1}} \right) du + O((\log f(y))^{\lambda - \delta}). \]
On the other hand, by (2.2.1) and since \( j_\lambda(.) \ll 1 \), we obtain:
\[ R \ll (\log f(y))^{\lambda - \delta}. \sum_{f(y) < f(p) \leq cx} \frac{1}{f(p)} \ll (\log f(y))^{\lambda - \delta} \lambda \log(v) \ll (\log f(y))^{\lambda - \delta}. \]
Hence, the expression (2.4) is

\[ F_f(x, y) = F_f(x) - \lambda v^\lambda (\log f(y))^{\lambda} \int_1^v \frac{j_\lambda(u - 1)}{u^{\lambda+1}} du + O((\log f(y))^{\lambda-\delta}). \tag{P.1.3} \]

Finally, using proposition 0 and (2.1.6), the expression (P.1.3) is equal to

\[ F_f(x, y) = A(f)(\log f(y))^{\lambda} j_\lambda(v)[1 + O((\log f(y))^{-\delta})] \]

and (R) is verified of \( k + 1 \). This proves the proposition.

**Proof of Proposition 2:**

We deduce from proposition 1 that if \( v \in [0, \lambda + 2] \)

\[ F_f(x, y) = A(f)(\log f(y))^{\lambda} j_\lambda(v)[1 + O((\log f(y))^{-\delta})]. \tag{P.2.1} \]

For \( v \geq \lambda \), define \( \Delta(v, y) \) by the following formula

\[ F_f(x, y) = A(f)(\log f(y))^{\lambda} j_\lambda(v)[1 + \Delta(v, y)] \tag{P.2.2} \]

Let

\[ \Delta^*(v, y) = \sup_{\lambda+1 \leq v' \leq v} |\Delta(v', y)| \]

then we have by (P.2.1)

\[ \Delta^*(v, y) \ll (\log f(y))^{-\delta}, \quad \lambda < v \leq \lambda + 2. \tag{P.2.3} \]

Our goal here is to prove

\[ \Delta^*(v, y) \ll \frac{\log v}{(\log f(y))^{\delta}} \tag{P.2.4} \]

uniformly for \( v \geq \lambda + 2 \) and for all sufficiently large \( y \) to prove the proposition. First, we return to the inequality in Lemma 6.

By (P.2.2), for \( v \geq \lambda + 2 \), we have

\[ \left( \log cx \right) \{A(f)(\log f(y))^{\lambda} j_\lambda(v)[1 + \Delta(v, y)]\} \leq \left( \log c \right) A(f)(\log f(y))^{\lambda} j_\lambda(v)[1 + \Delta(v, y)] \]

\[ + \int_{\frac{v}{2}}^{\frac{v}{2}} \frac{F_f(t, y)}{t} dt + \sum_{1 \leq f(p^k) \leq cx \atop p \leq y, k \geq 1} A(f)(\log f(y))^{\lambda} j_\lambda(v) \frac{\log f(p^k)}{\log f(y)} \frac{[1 + \Delta(v - \frac{\log f(p^k)}{\log f(y)}, y)] - \log f(p^k)}{f(p^k)} \tag{P.2.5} \]
The sum on the right of (P.2.5) is equal, with the notation \(v_p = \frac{\log f(y)}{\log f(y)}\) and \(v_{p^k} = \frac{\log f(p^k)}{\log f(y)}\), \(k \geq 2\), to

\[
\sum_{1 \leq f(p^k) \leq cx \atop p \leq y; k \geq 1} A(f)(\log f(y))^{\lambda} j_\lambda(v - \frac{\log f(p^k)}{\log f(y)}).[1 + \Delta(v - \frac{\log f(p^k)}{\log f(y)}, y)].\frac{\log f(p^k)}{f(p^k)}
\]

\[
= \sum_{1 \leq f(p) \leq cx \atop p \leq y} A(f)(\log f(y))^{\lambda} j_\lambda(v - v_p).[1 + \Delta(v - v_p)].\frac{\log f(p)}{f(p)}
\]

\[
+ \sum_{1 \leq f(p^k) \leq cx \atop p \leq y; k \geq 2} A(f)(\log f(y))^{\lambda} j_\lambda(v - v_{p^k}).[1 + \Delta(v - v_{p^k})].\frac{\log f(p^k)}{f(p^k)}.
\]

We consider the integral in (P.2.5), namely

\[
\int_{\frac{f(y)}{c}}^{x} \frac{F_t(t, y)}{t} dt = \int_{\frac{f(y)}{c}}^{\frac{f(y)}{c} + 1} \frac{F_t(t)}{t} dt + \int_{\frac{f(y)}{c} + 1}^{x} \frac{F_t(t, y)}{t} dt.
\]

By proposition 0, the first integral on the right is

\[
\frac{C(f)}{\lambda + 1}(\log f(y))^{\lambda+1}.[1 + O((\log f(y))^{-\delta})].
\]

Using proposition 1 and the change of variables \(u = \frac{\log et}{\log f(y)}\), the portion of the second integral that corresponds to \(\frac{f(y)}{c} \leq t \leq \frac{(f(y))^{\lambda+2}}{c}\) is equal to

\[
A(f)(\log f(y))^{\lambda+1}\int_{1}^{\lambda+2} j_\lambda(u)du + O((\log f(y))^{\lambda+1-\delta})
\]

and the remaining portion contributes, with the notation in (P.2.2)

\[
= A(f)(\log f(y))^{\lambda+1}\int_{\lambda+2}^{v} j_\lambda(u).[1 + \Delta(u, y)]du.
\]

Hence

\[
\int_{\frac{f(y)}{c}}^{x} \frac{F_t(t, y)}{t} dt = \frac{C(f)}{\lambda + 1}(\log f(y))^{\lambda+1} + A(f)(\log f(y))^{\lambda+1}.\left[\int_{1}^{\lambda+2} j_\lambda(t)dt + \int_{\lambda+2}^{v} j_\lambda(t) \cdot \Delta(t, y) dt\right]
\]

\[
+ O((\log f(y))^{\lambda+1-\delta}) ; \quad C(f) = \frac{A(f)}{\Gamma(\lambda + 1)}.
\]
And, we deduced from the above, that the inequality (P.2.5) is

\[
1 + \Delta(v, y) \leq [1 + \Delta(v, y)] \left[ \frac{\log c}{\log cx} + \frac{1}{\lambda(v) \log cx} \sum_{1 \leq f(p) \leq f(y)} \frac{\log f(p)}{f(p)} j_\lambda(v - v_p).[1 + \Delta(v - v_p, y)] \right]
\]

\[
+ \frac{1}{\lambda(v) \log cx} \sum_{1 \leq f(p) \leq f(y)} \frac{\log f(p)}{f(p_k)} j_\lambda(v - v_p).[1 + \Delta(v - v_p, y)]
\]

\[
+ \frac{\log f(y)}{(\lambda + 1)\Gamma(\lambda + 1)(\log cx)j_\lambda(v)} + \frac{\log f(y)}{(\log cx)j_\lambda(v)} \int_1^v j_\lambda(t) \, dt + \frac{\log f(y)}{(\log cx)j_\lambda(v)} \int_1^v j_\lambda(t) \Delta(t, y) \, dt
\]

\[
+ O\left( \frac{(\log f(y))^{1-\delta}}{j_\lambda(v) \log cx} \right).
\]

(P.2.6)

The second sum on the right of (P.2.6) is at most

\[
\frac{1 + \Delta^*(v, y)}{\log cx}
\]

by (Ω2). And using Lemma 7 (θ = 1), the first sum is equal to

\[
\frac{\lambda(\log f(y))}{j_\lambda(v) \log cx} \int_0^1 j_\lambda(v - t) \, dt + O\left( \frac{(\log f(y))^{1-\delta}}{j_\lambda(v) \log cx} \right)
\]

\[
+ \frac{1}{\lambda(v) \log cx} \sum_{1 \leq f(p) \leq f(y)} \frac{\log f(p)}{f(p)j_\lambda(v - v_p)} \Delta(v - v_p, y).
\]

We remarks that \( \int_0^1 j_\lambda(v - t) \, dt = \int_{v-1}^v j_\lambda(t) \, dt \), hence, with (2.1.7), the inequality (P.2.6) simplifies to

\[
\Delta(v, y) \leq \frac{1}{j_\lambda(v) \log cx} \sum_{1 \leq f(p) \leq f(y)} \frac{\log f(p)}{f(p)j_\lambda(v - v_p)} \Delta(v - v_p, y) + \frac{1}{v \lambda(v)} \int_1^v j_\lambda(t) \Delta(t, y) \, dt
\]

\[
+ O\left( \frac{1 + \Delta^*(v, y)}{v \log f(y)^{1-\delta}} \right).
\]

(P.2.7)

By vertue of Lemma 7 (θ = 1/2), the sum of the right of (P.2.7) that correspond to \( 1 \leq f(p) \leq (f(y))^{1/2} \) contribute at most

\[
\frac{\Delta^*(v, y)}{v j_\lambda(v)} \left[ \lambda \int_0^{\frac{1}{2}} j_\lambda(v - t) \, dt + O((\log f(y))^{-\delta}) \right]
\]
(since \(0 < \frac{\log f(p)}{\log f(y)} \leq \frac{1}{2}\)) and the remaining term contribute at most

\[
\frac{\Delta^*(v - \frac{1}{2}, y)}{v \lambda(v)} \int_{\frac{1}{2}}^1 j_\lambda(v - t) dt + O((\log f(y))^{-\delta})
\]

(since \(\frac{1}{2} < \frac{\log f(p)}{\log f(y)} \leq 1\)).

We consider the integral in (P.2.7). We have

\[
\frac{1}{v \lambda(v)} \int_{\lambda+2}^v j_\lambda(t) \Delta(t, y) dt \leq \frac{1}{v \lambda(v)} \left[ \int_1^{v-1} j_\lambda(t) \Delta(t, y) dt + \int_{v-1}^v j_\lambda(t) \Delta(t, y) dt \right]
\]

\[
\leq \frac{\Delta^*(v - 1, y)}{v \lambda(v)} \int_1^{v-1} j_\lambda(t) dt + \frac{\Delta^*(v, y)}{v \lambda(v)} \int_{v-1}^v j_\lambda(t) dt
\]

\[
\leq \frac{\Delta^*(v - \frac{1}{2}, y)}{v \lambda(v)} \int_1^{v-1} j_\lambda(t) dt + \frac{\Delta^*(v, y)}{v \lambda(v)} \int_{v-1}^v j_\lambda(t) dt
\]

and thus

\[
|\Delta(v, y)| \leq \Delta^*(v, y) \cdot \frac{1}{v \lambda(v)} \int_{v-1}^v j_\lambda(t) dt + \alpha(v) + \Delta^*(v - \frac{1}{2}, y) \cdot \frac{1}{v \lambda(v)} \int_1^{v-1} j_\lambda(t) dt + \beta(v)
\]

\[
+ O\left(1 + \Delta^*(v, y) \cdot \frac{1}{v (\log f(y))^\delta}\right). \tag{P.2.8}
\]

where

\[
\alpha(v) = \frac{\lambda}{v \lambda(v)} \int_0^{\frac{1}{2}} j_\lambda(v - t) dt
\]

and

\[
\beta(v) = \frac{\lambda}{v \lambda(v)} \int_1^{\frac{1}{2}} j_\lambda(v - t) dt.
\]

We note that

\[
\alpha(v), \beta(v) \leq \frac{\lambda}{2v}, \quad v > 1
\]

by the monotonicity of \(j_\lambda(.)\).

Introduce

\[
\alpha_1(v) = \frac{1}{v \lambda(v)} \int_{v-1}^v j_\lambda(t) dt + \alpha(v)
\]
and note, by (2.1.7), that

\[
\begin{align*}
&= \left[ \frac{\lambda}{v \cdot j_\lambda(v)} \int_{v-1}^v j_\lambda(t)dt + \frac{1}{v \cdot j_\lambda(v)} \int_1^v j_\lambda(t)dt \right] - \left[ \frac{1}{v \cdot j_\lambda(v)} \int_{v-1}^v j_\lambda(t)dt + \alpha(v) \right] \\
&= \frac{\lambda}{v \cdot j_\lambda(v)} \int_{v-1}^v j_\lambda(t)dt + \frac{1}{v \cdot j_\lambda(v)} \int_1^v j_\lambda(t)dt \\
&= \beta(v) + \frac{1}{v \cdot j_\lambda(v)} \int_1^{v-1} j_\lambda(t)dt \leq 1 - \alpha_1(v).
\end{align*}
\]

Hence, (P.2.8) simplifies to

\[
|\Delta(v, y)| \leq \Delta^*(v, y) \alpha_1(v) + \Delta^*(v - \frac{1}{2}, y) [1 - \alpha_1(v)] + O\left( \frac{1 + \Delta^*(v, y)}{v \log f(y)^\delta} \right) \quad v \geq \lambda + 2. \tag{P.2.9}
\]

We claim next that, by (P.2.9)

\[
|\Delta(v, y)| \leq \frac{1}{2} [\Delta^*(v, y) + \Delta^*(v - \frac{1}{2}, y)] + O\left( \frac{1 + \Delta^*(v, y)}{v \log f(y)^\delta} \right) \tag{P.2.10}
\]

uniformly for \( v \geq \lambda + 2 \) and if \( y \) is sufficiently large. Indeed, we observe that

\[
\frac{1}{2} \left[ \Delta^*(v, y) + \Delta^*(v - \frac{1}{2}, y) \right] - \left[ \Delta^*(v, y) \alpha_1(v) + \Delta^*(v - \frac{1}{2}, y) (1 - \alpha_1(v)) \right]
\]

\[
= \left( \frac{1}{2} - \alpha_1(v) \right) (\Delta^*(v, y) - \Delta^*(v - \frac{1}{2}, y))
\]

and this quantity is positive; for

\[
(\Delta^*(v, y) - \Delta^*(v - \frac{1}{2}, y)) \geq 0
\]

by the monotonicity of \( \Delta^* \), and

\[
\alpha_1(v) = \frac{1}{v \cdot j_\lambda(v)} \int_{v-1}^v j_\lambda(t)dt + \alpha(v) \leq \frac{1}{v} + \frac{\lambda}{2v} \leq \frac{1}{2}, \quad v \geq (\lambda + 2).
\]

This proves (P.2.10).

In order to show that (P.2.4) holds for \( v \geq \lambda + 2 \), first suppose that \( v - \frac{1}{2} \leq v' \leq v \).

Let \( A \) denote the \( O \)-constant in (P.2.10). By the monotonicity of \( \Delta^* \) and since \( \frac{1}{v'} \leq \frac{1}{v} \leq 1 - (2v)^{-1} \leq \frac{4}{3} \frac{1}{v} \) for \( v \geq (\lambda + 2) \) greater than 2

\[
\left| \Delta(v', y) \right| \leq \frac{1}{2} [\Delta^*(v', y) + \Delta^*(v' - \frac{1}{2}, y)] + A \left[ \frac{1 + \Delta^*(v, y)}{v \log f(y)^\delta} \right]
\]
\[ \leq \frac{1}{2}[\Delta^*(v, y) + \Delta^*(v - \frac{1}{2}, y)] + \frac{4}{3}A[\frac{1 + \Delta^*(v, y)}{v(\log f(y))^\delta}] \].

Now, if \((\lambda + 1) \leq v' \leq (v - \frac{1}{2})\), then
\[
|\Delta(v', y)| \leq \Delta^*(v - \frac{1}{2}, y) \leq \frac{1}{2}[\Delta^*(v, y) + \Delta^*(v - \frac{1}{2}, y)] + 4A\left(1 + \Delta^*(v, y)\right) \frac{1}{v(\log f(y))^\delta}.
\]
and it follows, by taking the on the right of \((P.2.10)\), that , uniformly for \(v \geq \lambda + 2\)
\[
\Delta^*(v, y) = \sup_{\lambda+1 \leq v' \leq v} |\Delta(y, v')| \leq \frac{1}{2}[\Delta^*(v, y) + \Delta^*(v - \frac{1}{2}, y)] + 4A\left(1 + \Delta^*(v, y)\right) \frac{1}{v(\log f(y))^\delta}.
\]
After rearranging terms we arrive at the inequality
\[
\Delta^*(v, y) \leq \Delta^*(v - \frac{1}{2}, y) + \frac{8}{3}A\left(1 + \Delta^*(v, y)\right) \frac{\log v}{v(\log f(y))^\delta}
\]
which we iterate to get
\[
\Delta^*(v, y) \leq \Delta^*(v_0, y) + \frac{8}{3}A\left(1 + \Delta^*(v, y)\right) \frac{\log v}{v(\log f(y))^\delta}
\]
where
\[
\lambda + 3/2 \leq v_0 \leq \lambda + 2
\]
By \((P.2.3)\), there exists \(A_1 > 0\) such that
\[
\Delta^*(v_0, y) \leq \frac{A_1}{(\log f(y))^\delta}
\]
and thus
\[
\Delta^*(v, y) \leq A^*[1 + \Delta^*(v, y)] \frac{\log v}{(\log f(y))^\delta}, \quad v \geq (\lambda + 2)
\]
where \(A^* = \sup(\frac{8}{3}A, A_1)\).
Hence
\[
\Delta^*(v, y)[1 - A^* \frac{\log v}{(\log f(y))^\delta}] \leq A^* \frac{\log v}{(\log f(y))^\delta}, \quad v \geq (\lambda + 2).
\]
With the condition \(1 \leq v \leq \exp\left(\frac{C_0(\log f(y))^\delta}{\log f(y)^\delta}\right)\), we have
\[
[1 - A^* \frac{\log v}{(\log f(y))^\delta}] \geq 1 - \frac{A^*}{C_0}
\]
and therefore, for \(C_0 > A^*\):
\[
\Delta^*(v, y) \ll \frac{\log v}{(\log f(y))^\delta}.
\]
This proves the theorem.
4 Proof of theorem 2

Lemma A:

Let $f \in \xi_{\lambda,\delta}$, then

$$S_f(x, y) \leq D \frac{x F_f(x, y)}{\log ecx}; \quad x > 1$$

where $D$ is a suitable constant.

**Proof:**

All we need here to prove the claim is the following weak consequence of $(\Omega_1)$ and $(\Omega_2)$:

There exists two constant $a > 0$ and $b > 0$ such that:

$$\sum_{f(p) \leq z} \log f(p) \leq az; \quad z \geq 1 \tag{A.1}$$

and

$$\sum_{(p, k) \geq 2} \frac{\log f(p^k)}{f(p^k)} \leq b. \tag{A.2}$$

We have the equation

$$(\log cx) S_f(x, y) = \sum_{f(n) \leq x} \log \left(\frac{cx}{f(n)}\right) + \sum_{f(n) \leq x} \sum_{P(n) \leq y} \log f(n) = T + M. \tag{A.3}$$

Since, $\log z \leq z - 1$ for $z > 0$, we have: $0 \leq T \leq cx F_f(x, y) - S_f(x, y)$.

Also,

$$M = \sum_{f(m)f(p^k) \leq x; k \geq 1} \sum_{(m, p)=1; P(mp) \leq y} \log f(p^k)$$

$$\leq \sum_{f(m)f(p) \leq x} \sum_{P(m) \leq y} \log f(p) + \sum_{f(m)f(p^k) \leq x} \sum_{P(mp) \leq y, k \geq 2} \log f(p^k)$$

$$\leq \sum_{f(m) \leq x} \sum_{f(p) \leq \frac{x}{P(m)}} \log f(p) + \sum_{f(p^k) \leq x; k \geq 2} S_f\left(\frac{x}{f(p^k)}; y\right) \log f(p^k)$$

$$\leq ax F_f(x, y) + \sum_{f(p^k) \leq x; k \geq 2} \frac{x}{f(p^k)} F_f\left(\frac{x}{f(p^k)}; y\right) \log f(p^k)$$
by \((A.1)\) and the obvious inequality \(S_f(x, y) \leq xF_f(x, y)\) applied in the second sum. 
Hence, by \((A.2)\)

\[
M \leq axF_f(x, y) + x \sum_{\frac{1}{e} \leq f(p) \leq 1} F_f(x, f(p), y) \log f(p) + x \sum_{1 < f(p) \leq x} F_f(x, f(p), y) \log f(p)
\]

\[
\leq axF_f(x, y) + bx F_f(x, y).
\]

Therefore

\[
S_f(x, y) \leq D \frac{x F_f(x, y)}{\log ecx}
\]

with \(D = c + a + b\).

**Proposition A:**

For \(f \in \xi_{\lambda, \delta}\) we have :

\[
(log cx)S_f(x, y) = \sum_{f(m) \leq x} \sum_{f(p) \leq \min(\frac{x}{f(m)}, f(y))} \log f(p) + O_{\lambda}(x(log f(y))^{\lambda}) ; v \geq 1.
\]

**Proof:**

We start again from the identity \((A.3)\). Suppose \(f(y) \leq cx\), we have

\[
M = \sum_{f(m)f(p) \leq x, (m,p)=1; P(m,p) \leq y} \log f(p) = \sum_{f(m) \leq x} \sum_{f(p) \leq \frac{x}{f(m)}, P(m) \leq y} \log f(p)
\]

\[
= \sum_{f(m) \leq x} \sum_{f(p) \leq \min(\frac{x}{f(m)}, f(y))} \log f(p) - \sum_{f(m)f(p) \leq x, p|m, P(mp) \leq y} \log f(p).
\]

This gives

\[
| (log cx)S_f(x, y) - \sum_{f(m) \leq x} \sum_{f(p) \leq \min(\frac{x}{f(m)}, f(y))} \log f(p) |
\]
Average of some multiplicative functions

\[ \leq T + \left| \sum_{f(m)f(p) \leq x, \ p|m; P(mp) \leq y} \log f(p) \right| + \left| \sum_{f(m)f(p^k) \leq x; k \geq 2, \ P(mp) \leq y} \log f(p^k) \right| = T + M' + M''. \]

We shall show that \( T, M' \) and \( M'' \) have order of \( \frac{x(\log f(y))^\lambda}{(\log x)^\delta} \).

We have by lemma A

\[ M' = \left| \sum_{f(l)f(p^{k+1}) \leq x; k \geq 1, \ (p,l)=1; P(pl) \leq y} \log f(p) \right| \]

\[ = \left| \sum_{f(p^{k+1}) \leq x, \ p \leq y; k \geq 1} \sum_{f(l) \leq \frac{x}{f(p^{k+1})}} \log f(p) \right| \]

\[ = \left| \sum_{f(p^k) \leq x, \ f(p) \leq f(y); k \geq 2} S_f\left( \frac{x}{f(p^k)}, y \right) \log f(p) \right| \]

\[ \ll xF_f(cx, y) \left| \sum_{f(p^k) \leq x, \ f(p) \leq f(y); k \geq 2} \frac{1}{f(p^k)} \log \left( \frac{cx}{f(p^k)} \right) \log f(p) \right| \]

\[ \ll xF_f(cx, y) \left[ \sum_{f(p^k) \leq \sqrt{x}, \ f(p) \leq f(y); k \geq 2} \frac{1}{f(p^k)} \log \left( \frac{cx}{f(p^k)} \right) \log f(p) + \sum_{\sqrt{x} < f(p^k) \leq x, \ f(p) \leq f(y); k \geq 2} \frac{1}{f(p^k)} \log f(p) \right] \]

by (2.2.3) and \( (\Omega_2) \).

The sum \( M'' \) is, for the same reasons, at most of order \( \frac{x(\log f(y))^\lambda}{(\log x)^\delta} \).
Finally, we estimate $T$. We see that, by Abel summation and lemma A

$$T = \int_{\frac{1}{c}}^{x} \log \left( \frac{x}{t} \right) d(S_f(t, y)) = \int_{\frac{1}{c}}^{x} S_f(t, y) \frac{dt}{t}\nless \int_{\frac{1}{c}}^{x} \frac{F_f(t, y)}{\log et} dt \nless F_f(x, y) \int_{\frac{1}{c}}^{x} \frac{dt}{\log et} \nless \frac{x F_f(x, y)}{\log x} \nless \frac{x (\log f(y))^\lambda}{(\log x)\delta}$$

This proves the lemma.

**Proposition B:**

Let $f \in \xi_{\lambda, \delta}$, then for $v \in ]0, 1[$

$$S_f(x, y) = S_f(x) = \sum_{f(n) \leq x} 1 = \lambda C(f) x (\log x)^{\lambda-1} + O(x (\log x)^{\lambda-\delta-1}).$$

**Proof:**

In a first step, we will show that for $v \in ]0, 1[$

$$S_f(x, y) = S_f(x) + O(x (\log x)^{\lambda-\delta-1}). \quad (B.1)$$

Indeed

$$S_f(x, y) = S_f(x) - \sum_{f(n) \leq x} 1 = S_f(x) - S_1.$$  

When we rearrange the sum $S_1$ following the largest prime divisor of $n$, we obtain

$$S_1 = \sum_{f(y) < f(p) \leq cx} S_f \left( \frac{x}{f(p)}, p \right) - \sum_{f(y) < p \leq cx} \sum_{f(m) \leq \frac{x}{f(p)}} \sum_{P(m) = p} 1 + \sum_{f(p^k) \leq cx; k \geq 2} \sum_{p > y} S_f \left( \frac{x}{f(p^k)} \right) = K + K' + K''.$$

For $v \in ]0, 1]$, $K = K' = 0$. On the other hand, by lemma A, proposition 0 and $(\Omega_2)$, $K''$ is at most

$$\sum_{f(p^k) \leq cx} \sum_{k \geq 2} \frac{x}{f(p^k) \log \left( \frac{cx}{f(p^k)} \right)} F_f \left( \frac{x}{f(p^k)} \right) \ll x (\log (cx))^{\lambda-\delta-1}$$

This proves $(B.1)$. 
Using (B.1), (Ω₁) and proposition 0, the equation established in proposition A becomes

\[
[S_f(x) + O(x(\log x)^{λ-δ-1})] \log cx = \sum_{f(m) ≤ x \atop P(m) ≤ y} \left[ \frac{x}{f(m)} + O\left( \frac{x}{f(m)} (\log(\frac{ecx}{f(m)})^{-δ}) \right) \right] + O\left( \frac{x F_f(x)}{(\log x)^{δ}} \right)
\]

\[
= λxF_f(x) + O(x(\log x)^{λ-δ}) + O(x \sum_{f(m) ≤ x \atop P(m) ≤ y} \frac{1}{f(m)} (\log(\frac{ecx}{f(m)})^{-δ}) ; v ∈ [0,1]. \tag{B.2}
\]

Now, let \( R \) denote the error term on the right of (B.2), and it suffices to show that \( R ≪ x(\log x)^{λ-δ} \).

Indeed, we have

\[
\int_{1/2}^{1} \frac{dt}{(\log ct)^{δ}} ≥ \frac{x}{f(m)} - 1/2 ≫ \frac{x}{f(m)} (\log(\frac{ecx}{f(m)})^{-δ})
\]

hence

\[
R ≪ \sum_{f(m) ≤ x \atop P(m) ≤ y} \int_{1/2}^{1} \frac{dt}{(\log ct)^{δ}} ≪ \int_{1/2}^{cx} S_f(\frac{x}{t}, y) \frac{dt}{(\log ct)^{δ}}.
\]

By lemma A and proposition 0, \( R \) is at most of order

\[
R ≪ x \int_{1/2}^{cx} \frac{F_f(x)}{t \log(\frac{ecx}{t})} (\log ct)^{δ} ≪ x(\log x)^{λ-1} \int_{1/2}^{cx} \frac{dt}{(\log ct)^{δ}} ≪ x(\log x)^{λ-δ}.
\]

Finally, we deduce that

\[
S_f(x) = λC(f)x(\log x)^{λ-1} + O(x(\log x)^{λ-δ-1}).
\]

**Deduction of Theorem 2 from Theorem 1:**

The proof of theorem 2 is obtained by combining theorem 1, lemma A and proposition A.

From proposition A, we obtain

\[
(\log cx)S_f(x, y) = \sum_{f(m) ≤ x \atop f(p) ≤ f(y)} \sum_{P(m) ≤ y} \log f(p)
\]
\[
\sum_{\substack{f(m) \leq x \atop P(m) \leq y}} \sum_{f(p) \leq f(y)} \log f(p) = \sum_{\substack{f(m) \leq x \atop P(m) \leq y}} \{\lambda f(y) + O(f(y)(\log f(y))^{-\delta})\}
\]

and using lemma A, we have
\[
f(y)S_f(x, y) \ll x \frac{F_f(x, y)}{\log(\frac{x}{f(y)})} \ll x(\log f(y))^{\lambda-1}
\]

(since \(F_f(x, y) \ll (\log f(y))^{\lambda}\) and \(\log f(y) \leq \log(\frac{ex}{f(y)})\) for \(v \geq 2\)).

Also, by \(\Omega_1\), we obtain
\[
\sum_{\substack{f(m) \leq x \atop P(m) \leq y}} \sum_{f(p) \leq f(y)} \log f(p) = \sum_{\substack{f(m) \leq x \atop P(m) \leq y}} \{\lambda \frac{x}{f(m)} + O(\frac{x}{f(m)}(\log(\frac{ex}{f(m)})^{-\delta}))\}
\]

\[
= \lambda x \{F_f(x, y) - F_f(\frac{x}{f(y)}, y)\} + O(\sum_{\substack{f(m) \leq x \atop P(m) \leq y}} \frac{x}{f(m)}(\log(\frac{ex}{f(m)})^{-\delta})).
\]

Now, we will show that the above error term, is at most
\[
\int_{1/2}^{f(y)} S_f(\frac{x}{t}, y) \frac{dt}{(\log et)^{\delta}}.
\]

Indeed, we have
\[
\int_{1/2}^{f(m)} \frac{dt}{(\log et)^{\delta}} \geq \left(\frac{x}{f(m)} - 1/2\right)(\log(\frac{ex}{f(m)})^{-\delta})
\]

and for \(1 \leq \frac{x}{f(m)} < f(y)\)
\[
\left(\frac{x}{f(m)} - 1/2\right) = \frac{x}{f(m)}(1 - \frac{1/2}{\frac{x}{f(m)}}) \geq \frac{x}{f(m)}(1 - 1/2)
\]
hence
\[ \frac{x}{f(m)} \ll \left( \frac{x}{f(m)} - 1/2 \right). \]

Therefore
\[ \frac{x}{f(m)} (\log \left( \frac{ex}{f(m)} \right)^{-\delta}) \ll \int_{1/2}^{\tau(m)} \frac{dt}{(\log et)^\delta}. \]

Thus, we get
\[
\sum_{\tau(y) < f(m) \leq x \atop P(m) \leq y} \frac{x}{f(m)} (\log \left( \frac{ex}{f(m)} \right)^{-\delta}) \ll \sum_{\tau(y) < f(m) \leq x \atop P(m) \leq y} \int_{1/2}^{\tau(m)} \frac{dt}{(\log et)^\delta} \ll \int_{1/2}^{f(y)} S_f(x, y) \frac{dt}{(\log et)^\delta},
\]

Using lemma A, the integral above it is, in turn, at most of order
\[
\int_{1/2}^{f(y)} S_f(x, y) \frac{dt}{(\log et)^\delta} \ll x \int_{1/2}^{f(y)} \frac{F_f(x, y)}{t \log \left( \frac{ex}{t} \right) (\log t)^\delta} \ll x \frac{(\log f(y))^\lambda}{(\log f(y))^\delta} \int_{1/2}^{f(y)} \frac{dt}{t (\log et)^\delta}
\ll x \frac{(\log f(y))^\lambda}{(\log f(y))^{1-\delta}} \ll x (\log f(y))^{\lambda-\delta}, \quad v \geq 2.
\]

The main term on the right side of (1.2) is equal to
\[
\lambda A(f) x (\log f(y))^{\lambda \left[ j_\lambda(v) - j_{\lambda-1}(v) \right]} + O(x (\log f(y))^{\lambda-\delta} \log v)
\]
(by theorem 1).

Thus, from (1.1), we obtain
\[
(\log cx) S_f(x, y) = \lambda A(f) x (\log f(y))^{\lambda \left[ j_\lambda(v) - j_{\lambda-1}(v) \right]} + O(x (\log f(y))^{\lambda-\delta} \log v)
\]

which reduces, after division by \( \log cx \)
\[
S_f(x, y) = A(f) x (\log f(y))^{\lambda-1} \{\rho_\lambda(v) + O(\frac{\log v}{(\log f(y))^\delta})\}; \quad v \geq 2.
\]

Now suppose \( 1 < v < 2 \). Again (1.1) becomes by \( (\Omega_1) \) and proposition B
\[
S_f(x, y) \log cx = \lambda f(y) S_f(x, y) \left[ 1 + ((\log f(y))^{-\delta}) + \lambda x [F_f(x, y) - F_f(x, f(y)] \right] + O(x (\log f(y))^\lambda)
\]
\[ + O(\sum_{\tau(y) < f(m) \leq x \atop P(m) \leq y} \frac{x}{f(m)} (\log \left( \frac{x}{f(m)} \right)^{-\delta}) \); \quad v \in [1, 2]. \quad (1.3)\]
We note that the last error term on the right side is at most of the same order as the error term on the right side of (1.2).

Together with theorem 1 and proposition B, we now can deduce from (1.3) that

\[ S_f(x, y) \log cx = \lambda A(f)x(\log f(y))^\lambda [j_\lambda(v) - j_\lambda(v - 1)] + O(x(\log f(y))^\lambda - \delta)) ; v \in ]1.2] \]

and thus

\[ = A(f)x(\log f(y))^\lambda v\rho_\lambda(v) + O(x(\log f(y))^\lambda - \delta)) ; v \in ]1.2] \]

(since \( \rho_\lambda(v) = j'_\lambda(v) \)).

After division by \( \log cx \), we get

\[ S_f(x, y) = A(f)x(\log f(y))^\lambda - 1[\rho_\lambda(v) + O\left(\frac{\log v}{(\log f(y))^\delta}\right)] ; v \in ]1.2]. \]

This proves theorem 2.

References


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