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Asymptotic Behavior for Semi-Linear Wave Equation with Weak Damping

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Abstract

In this work we study the asymptotic behavior as $t \to \infty$ of the solutions for the initial boundary value problem associated to the semi-linear wave equation with weak damping.

Keywords: Semi-linear wave, weak damping, exponential decay, Lemma of Nakao

1 Introduction

We shall consider the initial boundary value problem associated to the damped semi-linear wave equation,

$$u'' - \Delta u + F(u) + \alpha u' = 0, \qquad (1)$$

in the cylinder $Q = \Omega \times (0, T)$, $0 \leq T < \infty$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, α is a real positive constant, F is a continuous real function with $sF(s) \geq G(s) \geq 0$, for all $s \in \mathbb{R}^n$ and G is a primitive of F. The asymptotic behavior as $t \to \infty$ of the classical solution of equation with $F(u) = u^3$ was studied by Sattinger [7], for small initial data, and Nakao [5] without any restrictions concerning the smallness of the initial data. The exponential decay in the energy norm of weak solutions was proved by Strauss [8] and Nakao [6].

When $F(u) = |u|^{\rho-1}u$, $\rho \geq 1$, the existence of global weak solutions was studied by Lions [3] and the asymptotic behavior was proved by Avrin [2]. Existence of global weak solutions when $\alpha = 0$ was proved by Strauss [9]. He also proved that the local energy decay for zero as $t \to \infty$. In this work we prove that the global weak solutions of equation (1) decay exponentially. For our goal we use the following result

Lemma 1.1 *Let $E(t)$ be a nonnegative function on $[0,\infty)$ satisfying*

$$\sup_{s\in[t,t+1]} E(s) \leq C_0(\, E(t) - E(t+1)\,)$$

where C_0 is a positive constant. Then we have

$$E(t) \leq Ce^{-w\,t} \quad \text{with} \quad w = \frac{1}{C_0+1}.$$

Proof. 1.1 *See page 748 of [6].*

2 Asymptotic behavior

We use the standard Lebesgue space and Sobolev space with their usual properties as in [1] and in this sense $|\cdot|$ and $||\cdot||$ denote the norm in L^2 and H_0^1 respectively.

Lemma 2.1 *For F Lipschitz, derivable except on a finite number of points with $sF(s) \geq G(s) \geq 0$, for all $s \in \mathbb{R}$, the solution of the initial boundary value problem associated to the equation (1) with initial data $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $G(u_0) \in L^1(\Omega)$ satisfies*

$$|u'(t)|^2 + ||u(t)||^2 \leq C\, e^{-\,w\,t},$$

for all $t \geq 1$, where C and w are positive constants.

Proof. 2.1 *If F is Lipschitz and derivable except on a finite number of points, then there exists a unique solution u of (1), see [9] in the class*

$$u \in L^\infty(0,T;H_0^1(\Omega)), \tag{2}$$
$$u' \in L^\infty(0,T;L^2(\Omega)), \tag{3}$$

satisfying

$$u'' - \Delta u + F(u) + \alpha u' = 0 \ \text{ in } \ L^2(0,T;L^2(\Omega)) = L^2(Q). \tag{4}$$

Taking the inner product in $L^2(\Omega)$ of (4) with u' we obtain

$$\frac{d}{dt}E(t) + 2\alpha|u'(t)|^2 = 0, \tag{5}$$

where

$$E(t) = |u'(t)|^2 + ||u(t)||^2 + 2\int_\Omega G(u(t))\,dx \;\; with \;\; G(s) = \int_0^s F(\xi)\,d\xi.$$

Integrating (5) from t to $t+1$, we get

$$\int_t^{t+1} |u'(s)|^2\,ds = \frac{1}{2\alpha}[E(t) - E(t+1)] \equiv D^2(t). \tag{6}$$

The mean value theorem for integrals applied in (6) implies that there exists

$$t_1 \in \left[t, t+\frac{1}{4}\right] \;\; and \;\; t_2 \in \left[t+\frac{3}{4}, t+1\right] \;\; such\ that \;\; |u'(t_i)| \leq 2D(t), \;\; t_i = 1,2.$$

The inner product in $L^2(\Omega)$ of (4) with u implies

$$\frac{d}{dt}(u'(t), u(t)) - |u'(t)|^2 + ||u(t)||^2 + (F(u(t)), u(t)) + \alpha(u'(t), u(t)) = 0.$$

Integrating t_1 to t_2 and applying Cauchy-Schwarz's inequality and Poincare's inequality, we get

$$\begin{aligned}\int_{t_1}^{t_2} [||u(s)||^2 + (F(u(s)), u(s))]ds \;\; &\leq \;\; C_0[|u'(t_1)|||u(t_1)|| + |u'(t_2)|||u(t_2)||] \\ &+ \;\; \int_{t_1}^{t_2} |u'(s)|^2\,ds + \alpha\int_{t_1}^{t_2} |u'(s)|||u(s)||\,ds\end{aligned}$$

where C_0 denote Poincare's constant in Ω. Using $sF(s) \geq G(s) \geq 0$, $|u'(t_i)| \leq 2D(t)$, $t_i = 1,2$, and (6) we have

$$\int_{t_1}^{t_2} [||u(s)||^2 + (F(u(s)), u(s))]ds \;\; \leq \;\; 8C_0 D(t)\sqrt{E(t)} + (2 + \alpha^2 C_0^2)D(t)^2 \equiv H^2(t),$$

from where follows

$$\int_{t_1}^{t^2} E(s)ds \leq D^2(t) + H^2(t),$$

and by mean value theorem for integrals, there exists $t^* \in [t_1, t_2]$ such that

$$E(t^*) \leq 2D^2(t) + H^2(t). \tag{7}$$

Integrating (5) from t to t^*, using (6) and (7) follows that

$$E(t) \leq E(t^*) + 2\alpha + \int_t^{t+1} |u'(s)|^2\,ds \leq C_1 D^2(t) = \frac{C_1}{2\alpha}[E(t) - E(t+1)],$$

where $C_1 = 4 + (32+\alpha^2)C_0^2 + 2\alpha$. *Then*

$$\operatorname*{ess\,sup}_{s\in[t,t+1]} E(s) = E(t) \leq \frac{C_1}{2\alpha}[E(t) - E(t+1)],$$

and finally by **Lemma 1.1** *follows that*

$$E(t) \leq C\, e^{-w\,t},$$

for all $t \geq 1$ *with* C *and* w *positive constants, and by defintion of* $E(t)$ *we have* $|u'(t)|^2 + ||u(t)||^2 \leq C\, e^{-w\,t}$.

Now we are in the position to present our principal result.

Theorem 2.2 *If* F *is continuous with* $sF(s) \geq G(s) \geq 0$, *for all* $s \in \mathbb{R}$ *then the solution of the initial boundary value problem associated to the equation (1) with initial data* $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ *and* $G(u_0) \in L^1(\Omega)$ *satisfies*

$$|u'(t)|^2 + ||u(t)||^2 \leq C\, e^{-w\,t},$$

for all $t \geq 1$, *where* C *and* w *are positive constants.*

Proof. 2.3 *For* F *continuous there exists a sequence* $(F_k)_{k\in\mathbb{N}}$ *with each* F_k *Lipschitiz and derivable except on a finite number of points satisfying* $sF_k(s) \geq G_k(s) \geq 0$, *for all* $s \in \mathbb{R}$ *where* $G_k(s) = \int_0^s F_k(\xi)\, d\xi$, *such that* $F_k \to F$ *uniformly on the bounded sets of* $\mathbb{R}$. *For each* $k \in \mathbb{N}$ *let* u_k *be the solution of (1) when we replace* F *by* F_k. *For* u *solution of (1),* u_k *satisfies (see [9])*

$$u_k \to u \;\; \text{weakly-star in } L^\infty(0,T; H_0^1(\Omega)), \tag{8}$$

$$u_k' \to u' \;\; \text{weakly-star in } L^\infty(0,T; L^2(\Omega)), \tag{9}$$

$$F_k(u_k) \to F(u) \;\; \text{weakly in } L^1(\Omega), \tag{10}$$

$$u_k'' - \Delta u_k + F_k(u_k) + \alpha u_k' = 0 \;\; \text{in } L^2(Q) \tag{11}$$

From **Lemma 2.1** *we get* $|u_k'(t)|^2 + ||u_k(t)||^2 \leq C\, e^{-w\,t}$, *for all* $t \geq 1$. *Let be* $t_0 \in [1,T]$, *then*

$$|u_k'(t_0)|^2 + ||u_k(t_0)||^2 \leq C\, e^{-w\,t_0}, \tag{12}$$

$$u_k(t_0) \to \lambda \;\; \text{weakly in } H_0^1(\Omega) \;\; \text{as } k \to \infty, \tag{13}$$

$$u_k'(t_0) \to \eta \;\; \text{weakly in } L^2(\Omega) \;\; \text{as } k \to \infty. \tag{14}$$

$$\tag{15}$$

Using (8), (9) and Arzelá-Ascoli's theorem we have $\lambda = u(t_0)$. *Now, we are going to prove that* $\eta = u'(t_0)$. *For this we consider* $\theta \in [t_0, T[$ *defined for* $\delta > 0$ *by*

$$\theta(t) = \begin{cases} 1 \;\; \text{if } t = t_0, \\ -\frac{1}{\delta}(t - t_0 - \delta) \;\; \text{if } t_0 \leq t \leq t_0 + \delta, \\ 0 \;\; \text{if } t \geq t_0 + \delta. \end{cases}$$

It follows from (11) that $(u_k'', v) + (u_k, v) + (F_k(u_k), v) + \alpha(u_k', v) = 0$ *for all* $v \in L^2(\Omega)$*, in* $L^2(0,T)$*. Now multiplying by* θ *and integrating from* t_0 *to* T *we get for all* $v \in L^2(\Omega)$

$$-(u_k'(t_0), v) + \frac{1}{\delta}\int_{t_0}^{t_0+\delta} (u_k'(t), v)dt + \int_{t_0}^{t_0+\delta} a(u_k(t), v)\theta(t)dt$$
$$+ \int_{t_0}^{t_0+\delta} (F_k(u_k(t)), v)\theta(t)dt + \alpha\int_{t_0}^{t_0+\delta} (u_k'(t), v)\theta(t)dt = 0,$$

and taking the limit as $k \to \infty$ *using (10) and (14) we obtain*

$$-(\eta, v) + \frac{1}{\delta}\int_{t_0}^{t_0+\delta} (u'(t), v)dt + \int_{t_0}^{t_0+\delta} a(u(t), v)\theta(t)dt$$
$$+ \int_{t_0}^{t_0+\delta} (F(u(t)), v)\theta(t)dt + \alpha\int_{t_0}^{t_0+\delta} (u'(t), v)\theta(t)dt = 0.$$

Now taking again the limit as $\delta \to \infty$ *we get* $-(\eta, v) + (u'(t_0), v) = 0$ *and then* $\eta = u'(t_0)$*. Therefore, the* $\liminf$ *as* $k \to \infty$ *in (12) implies*

$$|u'(t_0)|^2 + ||u(t_0)||^2 \le C\, e^{-w\, t_0}. \tag{16}$$

If we consider $\theta \in C^0[0,T]$ *defined for* $\delta > 0$ *by*

$$\theta(t) = \begin{cases} 1 \;\; \textit{if} \;\; 0 \le t \le T - \delta, \\ \frac{1}{\delta}(t - T + \delta) \;\; \textit{if} \;\; t - \delta \le t \le T, \\ 0 \;\; \textit{if} \;\; t = T, \end{cases}$$

and repeating the same process we prove that (16) holds for $t = T$*. Finally,*

$$|u'(t)|^2 + ||u(t)||^2 \le C\, e^{-w\, t_0}.$$

for all $t \ge 1$ *and the proof of theorem is complete.*

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