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On Some Identities and Generating Functions for *k*- Pell Numbers

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Abstract

We obtain the Binet's formula for *k*-Pell numbers and as a consequence we get some properties for *k*-Pell numbers. Also we give the generating function for *k*-Pell sequences and another expression for the general term of the sequence, using the ordinary generating function, is provided.

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1. Introduction

Some sequences of number have been studied over several years, with emphasis on studies of well-known Fibonacci sequence (and then the Lucas sequence) that is related to the golden ratio. Many papers and research work are dedicated to Fibonacci sequence, such as the work of Hoggatt, in [15] and Vorobiov, in [13], among others and more recently we have, for example, the works of Caldwell *et al.* in [4], Marques in [7], and Shattuck, in [11]. Also

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relating with Fibonacci sequence, Falcőn *et al.*, in [14], consider some properties for *k*-Fibonacci numbers obtained from elementary matrix algebra and its identities including generating function and divisibility properties appears in the paper of Bolat *et al.*, in [3]. Other sequence, also important, is the sequence of Pell numbers defined by the recursive recurrence given by $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$, with $P_0 = 0$ and $P_1 = 1$. This sequence has been studied and some its basic properties are known (see, for example, the study of Horadam, in [2]). In [10], we find the matrix method for generating this sequence and comparable matrix generators have been considered by Kalman, in [6], by Bicknell, in [12], for the Fibonacci and Pell sequences. From this sequence, we obtain some types of other sequences namely, Pell-Lucas and Modified Pell sequences and also Dasdemir, in [1], consider some new matrices which are based on these sequences as well as that they have the generating matrices. The Binet's formula is also well known for several of these sequences. Sometimes some basic properties come from this formula. For example, for the sequence of Jacobsthal number, Koken and Bozkurt, in [8], deduce some properties and the Binet's formula, using matrix method. In [9], Yilmaz *et al*. study some more properties related with *k*- Jacobsthal numbers.

According Jhala *et al.*, in [5], we consider, in this paper, the *k*-Pell numbers sequence and many properties are proved by easy arguments for the *k*-Pell number.

2. The *k*-Pell Number and some identities

For any positive real number *k*, the *k*-Pell sequence say $(P_{k,n})_{n \in \mathbb{N}}$ is defined recurrently by

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \text{ for } n \geq 1, \tag{1}$$

with initial conditions given by,

$$P_{k,0} = 0, \; P_{k,1} = 1. \tag{2}$$

Next we find the explicit formula for the term of order *n* of the *k*-Pell numbers sequence using the well-known results involving recursive recurrences. Consider the following characteristic equation, associated to the recurrence relation (1),

$$r^2 - 2r - k = 0, \tag{3}$$

with two distinct roots r_1 and r_2. Note that the roots of the equation (3) are $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$, where k is a real positive number. Since $\sqrt{1+k} > 1$, then $r_2 < 0$ and so, $r_2 < 0 < r_1$.
Also, we obtain that $r_1 + r_2 = 2$, $r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$.

As a curiosity, for $k = 1$, we obtain that $r_1 = 1 + \sqrt{2}$ is the silver ratio which is related with the Pell number sequence. Silver ratio is the limiting ratio of consecutive Pell numbers.

Proposition 1 (Binet's formula)
The nth k-Pell number is given by

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad (4)$$

where r_1, r_2 *are the roots of the characteristic equation* (3) *and* $r_1 > r_2$.

Proof: Since the equation (3) has two distinct roots, the sequence

$$P_{k,n} = c_1(r_1)^n + c_2(r_2)^n \quad (5)$$

is the solution of the equation (1). Giving to n the values $n = 0$ and $n = 1$ and solving this system of linear equations, we obtain a unique value for c_1 and c_2. So, we get the following distinct values, $c_1 = \frac{1}{2\sqrt{1+k}}$ and $c_2 = -\frac{1}{2\sqrt{1+k}} = -c_1$. Since $r_1 - r_2 = 2\sqrt{1+k}$, we can express c_1 and c_2, respectively by $c_1 = \frac{1}{r_1 - r_2}$ and $c_2 = -\frac{1}{r_1 - r_2}$. Now, using (5), we obtain (4) as required.

■

Proposition 2 (Catalan's identity)

$$P_{k,n-r}P_{k,n+r} - P_{k,n}^2 = (-1)^{n+1-r}k^{n-r}P_{k,r}^2 \quad (6)$$

Proof: Using the Binet's formula (4) and the fact that $r_1 r_2 = -k$, we get

$$\begin{aligned} P_{k,n-r}P_{k,n+r} - P_{k,n}^2 &= \left(\frac{r_1^{n-r} - r_2^{n-r}}{r_1 - r_2}\right)\left(\frac{r_1^{n+r} - r_2^{n+r}}{r_1 - r_2}\right) - \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)^2 \\ &= (-1)^{n+1}\, k^{n-r}\left(\frac{r_1^r - r_2^r}{r_1 - r_2}\right)^2 \\ &= (-1)^{n+1}\, k^{n-r}\, P_{k,r}^2, \end{aligned}$$

that is, the identity required. ■

Note that for $r = 1$ in Catalan's identity obtained, we get the Cassini's identity for the k-Pell numbers sequence. In fact, the equation (6), for $r = 1$, yields

$$P_{k,n-1}P_{k,n+1} - P_{k,n}^2 = (-1)^n k^{n-1} P_{k,1}^2$$

and using one of the initial conditions of this sequence, we proved the following result.

Proposition 3 (Cassini's identity)

$$P_{k,n-1}P_{k,n+1} - P_{k,n}^2 = (-1)^n k^{n-1} \quad (7)$$

■

The d'Ocagne's identity can also obtained using the Binet's formula as it was done by Jhala *et al*. in [5] for the k-Jacobsthal sequence. Hence we have

Proposition 4 (d'Ocagne's identity)

If $m > n$ *then* $P_{k,m}P_{k,n+1} - P_{k,m+1}P_{k,n} = (-1)^n k^n P_{k,m-n}$. (8)

Proof: Once more, using the Binet's formula (4), the fact that $r_1 r_2 = -k$ and $m > n$, we get

$$\begin{aligned} P_{k,n-1}P_{k,n+1} - P_{k,n}^2 &= \left(\frac{r_1^m - r_2^m}{r_1 - r_2}\right)\left(\frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}\right) - \left(\frac{r_1^{m+1} - r_2^{m+1}}{r_1 - r_2}\right)\left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right) \\ &= (r_1 r_2)^n \left(\frac{(r_1 - r_2)(r_1^{m-n} - r_2^{m-n})}{(r_1 - r_2)^2}\right) \\ &= (r_1 r_2)^n \left(\frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2}\right) \\ &= (-k)^n P_{k,m-n} \\ &= (-1)^n k^n P_{k,m-n}. \end{aligned}$$

■

Again using the Binet's formula (4) we obtain other property of the k-Pell sequence which is stated in the following proposition.

Proposition 5

$$\lim_{n\to\infty} \frac{P_{k,n}}{P_{k,n-1}} = r_1. \quad (9)$$

Proof: We have that

$$\lim_{n\to\infty} \frac{P_{k,n}}{P_{k,n-1}} = \lim_{n\to\infty} \left(\frac{r_1^n - r_2^n}{r_1 - r_2}\right)\left(\frac{r_1 - r_2}{r_1^{n-1} - r_2^{n-1}}\right) = \lim_{n\to\infty} \left(\frac{r_1^n - r_2^n}{r_1^{n-1} - r_2^{n-1}}\right). \quad (10)$$

Using the ratio $\frac{r_2}{r_1}$ and since $\left|\frac{r_2}{r_1}\right| < 1$, then $\lim_{n\to\infty} \left(\frac{r_2}{r_1}\right)^n = 0$. Next we use this fact writing (10) with an equivalent form using this ratio, obtaining

$$\lim_{n\to\infty} \frac{P_{k,n}}{P_{k,n-1}} = \lim_{n\to\infty} \frac{1 - \left(\frac{r_2}{r_1}\right)^n}{\frac{1}{r_1} - \left(\frac{r_2}{r_1}\right)^n \frac{1}{r_2}} = \lim_{n\to\infty} \frac{1}{\frac{1}{r_1}} = r_1.$$

■

Also, we easily can show the following result using basic tools of calculus of limits and the (9).

Proposition 6

$$\lim_{n\to\infty}\frac{P_{k,n-1}}{P_{k,n}}=\frac{1}{r_1}. \qquad (11)$$

■

3. Generating functions for the *k*-Pell sequences

Next we shall give the generating functions for the *k*-Pell sequences. We shall write the *k*-Pell sequence as a power series where each term of the sequence correspond to coefficients of the series and from that fact, we find the generating function. Let us consider the *k*-Pell sequences $\left(P_{k,n}\right)_n$ for any positive integer *k*. By definition of ordinary generating function of some sequence, considering this sequence, the ordinary generating function associated is defined by

$$G(P_{k,n};x)=\sum_{n=0}^{\infty}P_{k,n}x^n=P_{k,0}+P_{k,1}x+\ P_{k,2}x^2+\cdots+P_{k,n}x^n+\ldots \qquad (12)$$

Using the initial conditions, we get

$$G(P_{k,n};x)=x+\ \sum_{n=2}^{\infty}P_{k,n}x^n. \qquad (13)$$

Now from (1) we can write (13) as follows

$$G(P_{k,n};x)=x+\ \sum_{n=2}^{\infty}(2P_{k,n-1}+kP_{k,n-2})x^n. \qquad (14)$$

Consider the right side of the equation (14) and doing some calculations, we obtain that

$$x+\sum_{n=2}^{\infty}(2P_{k,n-1}+kP_{k,n-2})x^n=x+2\sum_{n=2}^{\infty}P_{k,n-1}x^n+k\sum_{n=2}^{\infty}P_{k,n-2}x^n$$

$$=x+2x\sum_{n=2}^{\infty}P_{k,n-1}x^{n-1}+kx^2\sum_{n=2}^{\infty}P_{k,n-2}x^{n-2}\ . \qquad (15)$$

Consider that $j=n-2$ and $p=n-1$. Then (15) can be written by

$$x+2x\sum_{p=0}^{\infty}(P_{k,p}x^p-1)+kx^2\sum_{j=0}^{\infty}P_{k,j}x^j$$

$$=\ x+2x\sum_{p=0}^{\infty}P_{k,p}x^p+kx^2\sum_{j=0}^{\infty}P_{k,j}x^j-2x$$

$$= -x + 2x\sum_{p=0}^{\infty} P_{k,p}x^p + kx^2\sum_{j=0}^{\infty} P_{k,j}x^j. \qquad (16)$$

Therefore,

$$\sum_{n=0}^{\infty} P_{k,n}x^n = -x + 2x\sum_{n=0}^{\infty} P_{k,n}x^n + kx^2\sum_{n=0}^{\infty} P_{k,n}x^n,$$

which is equivalent to

$$\sum_{n=0}^{\infty} P_{k,n}x^n\,(1 - 2x - kx^2) = -x,$$

and then the ordinary generating function of the *k*-Pell sequence can be written as

$$\sum_{n=0}^{\infty} P_{k,n}x^n = \frac{-x}{1-2x-kx^2}\ . \qquad (17)$$

Recall that for a sequence $(a_n)_n$, if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L$, where L is a positive real number, then, considering the power series $\sum_{n=0}^{\infty} a_n x^k$, its radius of convergence R is equal to $\frac{1}{L}$. So, for the k-Pell sequence, using (9) and then (11) we know that she can be written as a power series with radius of convergence equal to $\frac{1}{r_1}$. Next we give another expression for the general term of the *k*-Pell sequence using the ordinary generating function.

Proposition 7

Let us consider, $p(x) = \sum_{n=0}^{\infty} P_{k,n}x^n$*, for* $x \in \left]-\frac{1}{r_1}, \frac{1}{r_1}\right[$*. Then we have that*

$$P_{k,n} = \frac{p^{(n)}(0)}{n!}, \qquad (18)$$

where $p^{(n)}(x)$ *denotes the nth order derivative of the function* $p(x)$*.*

Proof: We have from (12) that $p(0) = P_{k,0} = 0$ and

$$p'(x) = \sum_{n=1}^{\infty} nP_{k,n}\,x^{n-1} = 1 + \sum_{n=2}^{\infty} P_{k,n}\,x^{n-1}.$$

Also,

$$p^{(2)}(x) = \sum_{n=2}^{\infty} n(n-1)P_{k,n}\,x^{n-2} = 2.1.P_{k,2} + \sum_{n=3}^{\infty} n(n-1)P_{k,n}\,x^{n-2};$$

$$\vdots \quad \vdots \quad \vdots$$

$$p^{(l)}(x) = \sum_{n=l}^{\infty} n(n-1)\ldots(n-(l-1))P_{k,n}\,x^{n-l}.$$

So,

$p^{(l)}(x) =$
$l(l-1)\ldots\big(l-(l-1)\big)P_{k,l} + \sum_{n=l+1}^{\infty} n(n-1)\ldots(n-(l-1))P_{k,n}\, x^{n-l}$
and then we get $p^{(l)}(x) = l!\,P_{k,l} + \sum_{n=l+1}^{\infty} n(n-1)\ldots(n-(l-1))P_{k,n}\, x^{n-l}$. Therefore, $p^{(l)}(0) = l!\,P_{k,l}$ or $P_{k,l} = \frac{p^{(l)}(0)}{l!}$. Hence, for all $n \geq 1$, we have that $P_{k,n} = \frac{p^{(n)}(0)}{n!}$, as required.

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