On $e$ - Chaos

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Abstract

We define here $e$-chaos and study its interrelationships with $P$-chaos and Ruelle-Takens chaos. We show that a continuous self map $f$ on a compact metric space is $e$-chaotic ($P$-chaotic) need not imply the induced map $\bar{f}: K(X) \to K(X)$ is $e$-chaotic ($P$-chaotic) and vice-versa, where $K(X)$ is the space of all non-empty compact subsets of $X$ endowed with Hausdorff metric.

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1 Introduction

A numerous class of real problems are modeled by a discrete dynamical system $(X, f)$, where $X$ is a metric space and $f$ is a continuous self-map on $X$. The basic goal of the theory of discrete dynamical systems is to understand the nature of orbit $x, f(x), f^2(x), ..., f^n(x)$ of a point $x \in X$ as $n$ becomes large. The study of the orbit reveals how the points move in the base space $X$. Sometimes, it is not sufficient to know how points are moved in the base space $X$ but it is necessary to know how the subsets of $X$ are moved. This leads to the problem of analyzing the dynamics of the set-valued discrete dynamical systems. In [13], Román-Flores posed the following questions: What is the relationship between the dynamics of the individual movement and the dynamics of the collective movement? Since then many researchers have attempted to answer this question. For more details, one can refer [3], [8], [11], [14], [17].

The term chaos in connection with a map was first used by Li and Yorke without giving any formal definition [12]. Today there are various definitions of what it means for a system to be chaotic. A common idea of them is to show the complexity and unpredictability of behavior of the orbits of a system. Devaney’s definition of chaos is one of the most popular and widely
known definition of chaos. A map \( f : X \to X \) is called \textit{chaotic in the sense of Devaney} if \( f \) is transitive, the set of periodic points of \( f \) is dense in \( X \) and \( f \) has sensitive dependence on initial conditions \([7]\). In \([4]\), it is proved that the sensitive dependence on initial condition is redundant in the Devaney’s definition of chaos for infinite metric spaces. In \([2]\), authors have introduced the notion of \( P \)-chaos by replacing the transitivity condition in Devaney’s definition of chaos by pseudo orbit tracing property (POTP). A map \( f \) from a compact metric space \( X \) to itself is called \( P \)-chaotic if it has POTP and the set of periodic points of \( f \) is dense in \( X \). Positive topological entropy is considered as one of the form of chaos and it is observed that every \( P \)-chaotic map is Devaney chaotic and has positive topological entropy \([2]\).

Let \((X,d)\) be a metric space and let \((\mathcal{K}(X), H)\) be the space of all non-empty compact subsets of \( X \) endowed with the Hausdorff metric \( H \), where \( H(A, B) = \max\{d_{\text{dist}}(A, B), d_{\text{dist}}(B, A)\} \) and \( d_{\text{dist}}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b) \) for \( A, B \in \mathcal{K}(X) \). Note that the topology induced by the Hausdorff metric is same as the topology induced by the Vietoris topology on \( \mathcal{K}(X) \). We recall that a base for the Vietoris topology consists of all the sets of the form \( B(U_1, U_2, ..., U_n) = \{F \in \mathcal{K}(X) \mid F \subset \bigcup_{i=1}^{n} U_i, F \cap U_i \neq \emptyset, i = 1, 2, ..., n\} \), where \( U_1, U_2, ..., U_n \) are non-empty open sets in \( X \). For a continuous map \( f : X \to X \), the induced map \( \bar{f} : \mathcal{K}(X) \to \mathcal{K}(X) \) is defined by \( \bar{f}(A) = \{f(a) \mid a \in A\} \). It is observed that the continuity of the map \( f \) implies the continuity of the map \( \bar{f} \).

In this paper we introduce the notion of \( e \)-chaos and study its relation with \( P \)-chaos and Ruelle-Takens chaos. We also discuss whether \( f \) is \( e \)-chaotic (or \( P \)-chaotic) implies \( \bar{f} \) is \( e \)-chaotic (or \( P \)-chaotic) or not. It is well known that one of the conditions for detecting the presence of chaos is the sensitive dependence on initial conditions. This condition has been considered as the property of expansiveness of the dynamical system. There exists a close connection between the dynamics of expansive homeomorphisms and the topological properties of the underlying space. It can be seen that not all differentiable manifolds admit an expansive map. For example, if \( M \) is a one-dimensional compact connected manifold and \( f : M \to M \) is a homeomorphism, then \( f \) cannot be expansive. However, the situation changes for surfaces. A survey of spaces admitting expansive homeomorphism or otherwise can be found in \([5]\).

2 \( e \)-chaos

In this section we define the notion of \( e \)-chaos. A homeomorphism \( f : X \to X \) is said to be \textit{expansive} if there exists a constant \( c > 0 \) such that \( x \neq y \) in \( X \) implies \( d(f^n(x), f^n(y)) > c \) for some integer \( n \). The constant \( c \) is called an \textit{expansive constant} for \( f \).
**Definition 2.1.** A self homeomorphism \( f \) on a compact metric space \( X \) is said to be \( e \)-chaotic if it expansive and the set of all periodic points of \( f \) is dense in \( X \) [1].

**Lemma 2.2.** Let \( f : X \to X \) be a homeomorphism on a compact metric space \( X \). Then \( f \) is \( e \)-chaotic if and only if \( f^k \) is \( e \)-chaotic for \( k \neq 0 \).

**Proof.** Let \( P(f) \) denote the set of all periodic points of \( f \). Note that if \( P(f) \) is dense in \( X \), then \( \text{Cl}P(f) = \text{Cl}P(f^k) = X \) for all \( k \neq 0 \). The results now follows as \( f \) is expansive if and only if \( f^k \) is expansive for all \( k \neq 0 \) [1].

**Lemma 2.3.** Let \( f : X \to X \) be a homeomorphism on a compact metric space \( X \). Then \( f \) is \( e \)-chaotic is a topological property.

**Proof.** Follows as expansivity and density of periodic points are preserved under topological conjugacy.

**Theorem 2.4.** Let \( f : X \to X \) be a homeomorphism on a compact metric space \( (X,d) \). Then \( f \) is expansive implies \( f \) is expansive.

**Proof.** Let \( x, y \in X, x \neq y \). Then consider the compact sets \( A_1 = \{ x \} \) and \( A_2 = \{ y \} \) in \( K(X) \). Since \( f \) is expansive, there exist a constant \( c > 0 \) such that \( H(f^n(A_1), f^n(A_2)) > c \) for some integer \( n \). But \( H(f^n(A_1), f^n(A_2)) = d(f^n(x), f^n(y)) > c \). Thus \( f \) is expansive.

**Theorem 2.5.** Let \( f : X \to X \) be a homeomorphism on a metric space \( (X,d) \). If \( f \) is periodically dense and \( f^* \) is expansive then \( f^* \) is \( e \)-chaotic.

**Proof.** Follows since periodic density of \( f \) implies periodic density of \( f^* \) [3].

**Remark.** It is known that periodic density of \( f^* \) need not imply periodic density of \( f \) [3]. Thus \( f^* \) is \( e \)-chaotic need not imply \( f \) is \( e \)-chaotic. It is known that the shift map, \( \sigma \), on two symbols is expansive and its topological entropy, \( h(\sigma) \), is \( \log 2 \). By a result in [11], it follows that \( h(\sigma) = \infty \). That \( \sigma \) is not expansive follows as expansive maps on compact metric spaces have finite topological entropy. Thus \( f \) is expansive need not imply \( f^* \) is expansive.

In [14], it is proved that if \( X \) is a non-empty compact convex subset of a normed linear space \( E \) and if \( f : X \to X \) is a continuous map, then \( f^* \) has periodic density on \( K_c(X) \), the collection of all non-empty compact convex subsets of \( X \), implies that \( f \) has periodic density. Hence as a consequence we have the following result.

**Theorem 2.6.** Let \( X \) be a non-empty compact convex subset of a normed linear space \( E \) and let \( f : X \to X \) be a homeomorphism. Suppose \( f^* \) is expansive and has periodic density on \( K_c(X) \). Then \( f \) is \( e \)-chaotic.
3 e-chaos and P-chaos

In this section we study the relationship of P-chaotic maps and e-chaotic maps. Let \( f : X \to X \) be a continuous map on a metric space \((X, d)\). A sequence of points \( \{x_i | i \geq 0\} \) in \( X \) is said to be a \( \delta \)-pseudo-orbit for \( f \) if \( d(f(x_i), x_{i+1}) < \delta \) for each \( i \). A sequence \( \{x_i | i \geq 0\} \) is said to be \( \epsilon \)-traced by a point \( x \) in \( X \) if \( d(f^i(x), x_i) < \epsilon \) holds for each \( i \geq 0 \). A map \( f \) is said to have the pseudo orbit tracing property (POTP) if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that each \( \delta \)-pseudo-orbit for \( f \) is \( \epsilon \)-traced by some point of \( X \). A map \( f \) from a compact metric space \( X \) to itself is called \( P \)-chaotic if it has POTP and the set of periodic points of \( f \) is dense in \( X \).

Example 3.1. Let \( Y_k = \{0, 1, 2, ..., k-1\} \) and let \( Y_k^\mathbb{Z} = \prod_{-\infty}^{\infty} Y_k \). The shift homeomorphism \( \sigma : Y_k^\mathbb{Z} \to Y_k^\mathbb{Z} \) is defined by \( (\sigma(x))_i = x_{i+1} \), where \( x = (x_i)_{i \in \mathbb{Z}} \). The map \( \sigma \) is expansive, has POTP and has dense set of periodic points. Thus \( \sigma \) is both e-chaotic as well as \( P \)-chaotic.

Example 3.2. Let \( k = 2 \) in the Example 3.1 and let \( S = \{(x_i) \in Y_k^\mathbb{Z} | (x_i, x_{i+1}) \in C, i \in \mathbb{Z}\} \), where \( C = \{(0,0), (0,1), (1,1)\} \). Then \( \sigma \) restricted to \( S \) is expansive and has POTP, but \( S \) contains only two periodic points, which are in fact fixed points, namely \( x = (..., 0, 0, ..., y = (..., 1, 1, ...) \). Thus \( \sigma \) restricted to \( S \) is neither \( P \)-chaotic nor e-chaotic.

Example 3.3. Let \( X = \left\{ \frac{1}{n} | n \in \mathbb{N} \} \cup \{0\} \) and let \( f : X \to X \) be the identity map. Then \( f \) has POTP and \( \text{Per}(f) = X \). Thus \( f \) is \( P \)-chaotic. The map \( f \) being isometry is not expansive and hence not e-chaotic.

Remark. Above two examples show that expansivity and density of periodic points are independent. A pseudo-Anosov map is expansive but does not have POTP [6]. Thus e-chaotic map need not be \( P \)-chaotic.

Theorem 3.4. Let \( f : X \to X \) be a continuous map. If \( \overline{f} : \mathcal{K}(X) \to \mathcal{K}(X) \) has POTP then \( f \) has POTP.

Proof. Let \( \epsilon > 0 \) be given. Then since \( \overline{f} \) has POTP, there exists \( \delta > 0 \) such that each \( \delta \)-pseudo-orbit for \( \overline{f} \) is \( \epsilon \)-traced by some point of \( \mathcal{K}(X) \). Let \( \{x_i\} \) be a \( \delta \)-pseudo-orbit for \( f \). Then choose \( A_i = \{x_i\} \), which is a \( \delta \)-pseudo-orbit for \( \overline{f} \) and hence is \( \epsilon \)-traced by some point \( A \) of \( \mathcal{K}(X) \). If \( A \) is singleton then we are through, otherwise using compactness of \( X \), we can get \( x \in A \), which \( \epsilon \)-traces \( \{x_i\} \).

Remark. Periodic density of \( \overline{f} \) need not imply periodic density of \( f \) [3]. Thus \( \overline{f} \) is \( P \)-chaotic need not imply \( f \) is \( P \)-chaotic. However, if \( f \) is periodically dense and \( f \) has POTP then \( \overline{f} \) is \( P \)-chaotic.
The following example shows that a map \( f \) may have POTP but the induced map \( \overline{f} \) need not have POTP.

**Example 3.5.** Let \( X = \{ \frac{1}{n} | n \in \mathbb{N} \} \cup \{1 - \frac{1}{n} | n \in \mathbb{N} \} \) and let \( f : X \to X \) be defined by \( f(0) = 0, f(1) = 1 \) and for \( x \in X - \{0,1\} \), \( f(x) \) is defined to be an element of \( X \) immediate to right of \( x \). Then \( f \) has POTP, but \( \overline{f} \) does not have POTP. Let \( 0 < \epsilon < \frac{1}{6} \). Then we show that for every \( \delta > 0 \) there exists a \( \delta \)-pseudo orbit of \( \overline{f} \), which is not \( \epsilon \)-traced by any point of \( \mathcal{K}(X) \).

**Case (i):** For \( \delta \geq \frac{1}{2} \), consider \( A_n = \{ \frac{1}{n+1}, 1 - \frac{1}{n+1} \} \), \( n \in \mathbb{N} \). Then \( \{A_n | n \in \mathbb{N} \} \) is a \( \delta \)-pseudo orbit as \( H(f(A_1), A_2) = \frac{1}{3} \) and \( H(f(A_n), A_{n+1}) = \frac{\delta}{n+1} < \delta \), for all \( n \geq 2 \). If possible suppose there exists \( A \in \mathcal{K}(X) \), which \( \epsilon \)-traces \( \{A_n \} \). Then \( H(f^n(A), A_n) < \epsilon \) for all \( n \in \mathbb{N} \). Now \( H(f(A), A_1) < \epsilon \) implies that \( f(A) \subseteq N(\epsilon, A_1) \). This further implies that \( f(A) \subseteq N(\epsilon, f^2(A)) \), i.e., \( A_2 \subseteq N(\epsilon, \{\frac{2}{3}\}) \), which is not possible.

**Case (ii):** For \( 0 < \delta < \frac{1}{6} \), there exists \( n \in \mathbb{N} \) such that \( \frac{1}{n+1} < \delta < \frac{1}{6} \). The \( \delta \)-pseudo orbit \( A_1 = \{\frac{1}{2}\} \), \( A_2 = \{\frac{1}{n+1}, \frac{2}{3}\} \), \( A_3 = \{\frac{1}{n+1}, \frac{2}{3}, \frac{3}{5}\} \),..., is not \( \epsilon \)-traced by any point of \( \mathcal{K}(X) \). If possible suppose there exists \( A \in \mathcal{K}(X) \), which \( \epsilon \)-traces \( \{A_n \} \). Then \( H(f^n(A), A_n) < \epsilon \) for all \( n \in \mathbb{N} \). Now \( H(f(A), A_1) < \epsilon \) implies that \( f(A) \subseteq N(\epsilon, A_1) \) and hence \( f(A) \subseteq \{\frac{1}{5}, \frac{1}{3}, \frac{1}{\delta}\} \). Next, \( H(f^2(A), A_2) < \epsilon \) implies \( f^2(A) \subseteq N(\epsilon, A_2) \). Now combining the facts that \( f(A) \subseteq \{\frac{1}{5}, \frac{1}{3}, \frac{1}{\delta}\} \) and \( f^2(A) \subseteq N(\epsilon, A_2) \), we get \( f^2(A) \subseteq \{\frac{1}{7}, \frac{1}{5}\} \). Similar argument shows that \( f^3(A) = \{\frac{2}{5}\} \). Therefore \( f^2(A) = \{\frac{2}{5}\} \). Now, \( H(f^2(A), A_2) < \epsilon \) implies \( A_2 \subseteq N(\epsilon, f^2(A)) \subseteq \{\frac{5}{3}, \frac{7}{3}\} \), which is not possible as \( \frac{1}{n+1} < \frac{1}{\delta} \).

**Case (iii):** For \( \frac{1}{6} \leq \delta < \frac{1}{2} \), choose \( \delta \)-pseudo orbit \( A_1 = \{\frac{1}{2}\} \), \( A_2 = \{\frac{1}{5}, \frac{1}{3}\} \), \( A_3 = \{\frac{1}{5}, \frac{1}{3}, \frac{1}{\delta}\} \). Then \( \{A_n \} \) is not \( \epsilon \)-traced by any point of \( \mathcal{K}(X) \) can be argued similarly as in the previous two cases.

**Remark.** We recall that a continuous onto map \( f : X \to X \) is positively expansive if there exists a constant \( c > 0 \) such that \( x \neq y \) implies \( d(f^n(x), f^n(y)) > c \) for some \( n > 0 \). In [15], it is proved that for a positively expansive map on a compact metrizable space, the map \( f \) is open is equivalent to \( f \) has POTP. In [17], authors have proved that if \( f \) is a positively expansive open map on a compact metric space \( X \), then the induced map \( \overline{f} \) has POTP. In a way it gives a condition under which \( f \) has POTP implies \( \overline{f} \) has POTP. They have given an example of a map \( f \) which is positively expansive open map for which \( \overline{f} \) has POTP but \( \overline{f} \) is not positively expansive. Thus \( f \) is positively expansive need not imply \( \overline{f} \) is positively expansive.
4 e-chaos and Ruelle-Takens chaos

Let \( Y \) be an invariant subset of \( X \). Then the map \( f : X \to X \) is said to be \textit{chaotic on} \( Y \) \textit{in the sense of Ruelle-Takens} if (i) \( f|_Y : Y \to Y \) is one-sided topologically transitive and (ii) \( f|_Y \) has sensitive dependence on initial conditions. In [9], it is proved that if a homeomorphism \( f : X \to X \) of a compact metric space \( X \) with \( \dim X > 0 \) is continuum-wise expansive, then there is an \( f \)-invariant closed subset \( Y \) of \( X \) with \( \dim Y > 0 \) such that \( f \) is chaotic on \( Y \) in the sense of Ruelle-Takens. In this section we study the relation between e-chaos and Ruelle-Takens chaos.

We first recall the basic definitions. A homeomorphism \( f : X \to X \) is called \textit{continuum-wise expansive} if there exists a constant \( c > 0 \) such that whenever \( A \) is a nondegenerate subcontinuum of \( X \), then there is an integer \( n \in \mathbb{Z} \) such that \( \text{diam} f^n(A) \geq c \) [10]. Every expansive homeomorphism is continuum-wise expansive but the converse is not true [10]. A self map \( f \) of a compact metric space \( X \) is said to be \textit{one-sided topologically transitive} if there exists some \( x \in X \) with \( \{f^n(x) | n \geq k\} \) is dense in \( X \) for each \( k \geq 0 \). This is equivalent to the fact that whenever \( U \) and \( V \) are open sets of \( X \) there exists \( n \geq 1 \) with \( f^{-n}(U) \cap V \neq \emptyset \) [16]. In [9], it is proved that for compact metric spaces if \( f \) is one-sided topologically transitive then it is topologically transitive.

Since an expansive map is a continuum-wise expansive map, the following Theorem follows from Proposition 5.2 [10].

**Theorem 4.1.** Let \( f : X \to X \) be e-chaotic map on a continuum \( X \). Suppose for each natural number \( n \geq 1 \) and each proper closed subset \( E \) of \( X \) with \( \dim E > 0 \), \( f^n(E) \neq E \). Then \( f \) is topologically mixing.

**Remark.** For a perfect metric space, \( f \) is expansive implies that \( f \) has sensitive dependence on initial conditions. Also, \( f \) is topologically mixing implies \( f \) is transitive. Note that a mixing map need not be an expansive map. For example, an interval admits a mixing map which is not expansive. The map in Example 3.4 is expansive but not mixing.

**Theorem 4.2.** Let \( f : X \to X \) be e-chaotic map on a continuum \( X \). Suppose for each natural number \( n \geq 1 \) and each proper closed subset \( E \) of \( X \) with \( \dim E > 0 \), \( f^n(E) \neq E \). Then \( f \) is Ruelle-Takens chaotic on \( X \).

**Proof.** Follows from Theorem 4.1 and the above remark.

**Remark.** Ruelle-Takens chaotic maps need not be e-chaotic. For example any transitive map on an interval is Ruelle-Takens chaotic but is not e-chaotic as intervals do not admit expansive maps.
The following diagram summarizes the results obtained in the paper.

\[
\begin{array}{c}
\text{Ruelle-Takens chaos} \nsim \quad \text*e-chaos} \nsim \quad \text{Mixing} \\
\downarrow \quad \uparrow \\
\text{P-chaos}
\end{array}
\]

Let \((X, d, f)\) be a compact system. We have shown some connections and differences of the complexity of \((X, d, f)\) to that of its induced hyperspace system \((\mathcal{K}(X), H, \mathcal{T})\), especially on the aspects of \(e\)-chaos and \(P\)-chaos.

\[
\begin{array}{c|c|c}
\text{Notions of chaos} & f & \mathcal{T} \\
\hline
\text{e-chaos, P-chaos} & \nsim & \nsubseteq
\end{array}
\]

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**References**


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