

Hermite-Hadamard-like and Simpson-like Type Inequalities for Harmonically Convex Functions

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Abstract

In this paper, by setting up a generalized integral identity for differentiable functions, the author obtain some new upper bounds of Hermite-Hadamard type inequalities and new Simpson-like type inequalities, for differentiable harmonically convex functions.

Mathematics Subject Classification: 26A51, 26D15

Keywords: Hermite-Hadamard type inequality, Simpson type inequality, Hölder's inequality, Harmonically conexity

1 Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \rightarrow R$ be a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Hermite-Hadamard's inequalities for convex functions and geometrically convex functions have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1]-[14] and references therein.

Let us recall some definitions of several kinds of convex functions:

Definition 1. Let I be an interval in R . Then $f : I \rightarrow R$ is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \rightarrow R$ is said to be harmonically convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \quad (2)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

In [4], Imdat İşcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

Theorem 1.1. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a harmonically convex function on an interval I and $f \in L[a, b]$, where $a, b \in I$ with $a < b$.

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \quad (3)$$

Also, in [4], Imdat İşcan established some new Hermite-Hadamard type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

Theorem 1.2. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \lambda_1^{1-\frac{1}{q}} [\lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q]^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2} \ln\left(\frac{(a+b)^2}{4ab}\right), \\ \lambda_2 &= -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln\left(\frac{(a+b)^2}{4ab}\right), \\ \lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln\left(\frac{(a+b)^2}{4ab}\right) \\ &= \lambda_1 - \lambda_2. \end{aligned}$$

In [5], Imdat İşcan established the following theorems:

Theorem 1.3. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then we have the following inequality for $\lambda \in [0, 1]$:

$$\begin{aligned} & \left| \lambda \left(\frac{f(a) + f(b)}{2} \right) + (1 - \lambda) \left(\frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{2} \left\{ C_1^{1-\frac{1}{q}}(\lambda; a, b) [C_2(\lambda; a, b) |f'(a)|^q \right. \\ & \quad \left. + C_3(\lambda; a, b) |f'(b)|^q]^\frac{1}{q} + C_1^{1-\frac{1}{q}}(\lambda; a, b) \right. \\ & \quad \times [C_3(\lambda; a, b) |f'(a)|^q + C_2(\lambda; a, b) |f'(b)|^q]^\frac{1}{q} \left. \right\}, \end{aligned}$$

where

$$\begin{aligned} C_1(\lambda; u, v) &= \frac{1}{(v-u)^2} \\ &\times \left[-4 + \frac{\{\lambda(v-u) + 2u\}(3u+v)}{u(u+v)} + 2 \left(\frac{2u(u+v)}{(2u+\lambda(v-u))^2} \right) \right], \\ C_2(\lambda; u, v) &= \frac{1}{(v-u)^2} \\ &\times \left[\{\lambda(v-u) + 4u\} \ln \left(\frac{\{\lambda(v-u) + 2u^2\}}{2u(u+v)} \right) \right. \\ &\quad \left. - \frac{\{\lambda(v-u) + 2u\}(5u+3v)}{u+v} + 7u+v \right], \end{aligned}$$

and

$$C_3(\lambda; u, v) = C_1(\lambda; u, v) - C_2(\lambda; u, v), \quad u, v > 0.$$

Theorem 1.4. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically convex function on $[a, b]$ for $q > 1$, then we have the following inequality for $\lambda \in [0, 1]$:

$$\begin{aligned} & \left| \lambda \left(\frac{f(a) + f(b)}{2} \right) + (1 - \lambda) \left(\frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ & \leq \frac{ab(b-a)}{4} \frac{C_4^{\frac{1}{q}}(\lambda, p)}{\{(1-q)(1-2q)(b-a)^2\}^{\frac{1}{q}}} \\ & \quad \times \left\{ (C_5(q; a, b) |f'(a)|^q + C_6(q; a, b) |f'(b)|^q)^\frac{1}{q} \right. \\ & \quad \left. + (C_6(q; a, b) |f'(a)|^q + C_5(q; b, a) |f'(b)|^q)^\frac{1}{q} \right\}, \end{aligned}$$

where

$$C_4(\lambda, p) = \frac{\lambda^{1+p} + (1-\lambda)^{1+p}}{p+1},$$

$$C_5(q; u, v) = \left[\left(\frac{u+v}{2} \right)^{1-2q} \left\{ \frac{v-3u}{2} - q(v-u) \right\} + u^{2-2q} \right],$$

$$\begin{aligned} C_6(q; u, v) = & \left[\left(\frac{u+v}{2} \right)^{1-2q} \left\{ \frac{3v-u}{2} - q(v-u) \right\} \right. \\ & \left. + u^{1-2q} \left\{ u - 2v + 2q(v-u) \right\} \right], \quad u, v > 0 \end{aligned}$$

and

$$\frac{1}{p} + \frac{1}{q\alpha} = 1.$$

Definition 3. The hypergeometric function ${}_2F_1[a, b, c, x]$ is defined for $|x| < 1$ by the power series

$${}_2F_1[a, b, c, x] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \quad (4)$$

It is undefined if c equals a non-positive integer. Here $(q)_n$ is the Pochhammer symbol, which is defined by

$$(q)_n = \begin{cases} 1, & n = 0 \\ q(q+1) \cdots (q+n-1), & n > 0. \end{cases}$$

Definition 4. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt. \quad (5)$$

In this paper, we give some generalized inequalities connected with the left and right parts of (3), as a result of this, we obtain some generalized Hermite-Hadamard-like and Simpson-like type inequalities for differentiable harmonically convex functions by setting up an integral identity for differentiable functions.

2 Main results

In this paper, for the simplicity of notations, let us denote

$$\begin{aligned}
\mu_{11} &= \frac{\alpha\lambda}{a(b-a)} + \frac{1}{(b-a)^2} \ln \left(\frac{a}{A_{\lambda\alpha}} \right), \\
\mu_{12} &= \frac{\alpha\lambda + \alpha - 1}{(b-a)A_{1-\alpha}} + \frac{1}{(b-a)^2} \ln \left(\frac{A_{1-\alpha}}{A_{\lambda\alpha}} \right), \\
\mu_{13} &= \frac{(1-\alpha)A_{\lambda\alpha}}{a(b-a)A_{1-\alpha}} + \frac{1}{(b-a)^2} \ln \left(\frac{a}{A_{1-\alpha}} \right), \\
\mu_{21} &= \frac{\alpha\{(1-\alpha)\lambda(b-a) - b\}}{b(b-a)A_{1-\alpha}} + \frac{1}{(b-a)^2} \ln \left(\frac{b}{A_{1-\alpha}} \right), \\
\mu_{22} &= \frac{\alpha + \alpha\lambda - \lambda}{(b-a)A_{1-\alpha}} + \frac{1}{(b-a)^2} \ln \left(\frac{A_{1-\alpha}}{A_{1-\lambda+\alpha\lambda}} \right), \\
\mu_{23} &= \frac{\lambda(\alpha-1)}{b(b-a)} + \frac{1}{(b-a)^2} \ln \left(\frac{b}{A_{1-\lambda+\alpha\lambda}} \right), \\
\mu_{31} &= \frac{2\alpha\lambda}{(b-a)^2} + \frac{a + A_{\alpha\lambda}}{(b-a)^3} \ln \left(\frac{A_{\alpha\lambda}}{a} \right), \\
\mu_{32} &= \frac{(1-\alpha-\alpha\lambda)(a + A_{1-\alpha})}{(b-a)^2A_{1-\alpha}} + \frac{a + A_{\alpha\lambda}}{(b-a)^3} \ln \left(\frac{A_{\alpha\lambda}}{A_{1-\alpha}} \right), \\
\mu_{33} &= \frac{\alpha\lambda(a+b)}{a(b-a)^2} + \frac{b + A_{\alpha\lambda}}{(b-a)^3} \ln \left(\frac{a}{A_{\alpha\lambda}} \right), \\
\mu_{34} &= \frac{(\alpha\lambda + \alpha - 1)(b + A_{1-\alpha})}{(b-a)^2A_{1-\alpha}} + \frac{b + A_{1-\alpha\lambda}}{(b-a)^3} \ln \left(\frac{A_{1-\alpha}}{A_{\alpha\lambda}} \right), \\
\mu_{41} &= \frac{(1-\alpha)(3a + A_{\alpha-\alpha\lambda-1})}{(b-a)^2A_{1-\alpha}} + \frac{a + A_{\alpha\lambda}}{(b-a)^3} \ln \left(\frac{A_{1-\alpha}}{a} \right), \\
\mu_{42} &= \frac{(1-\alpha)\{\alpha a^2 + \alpha\lambda b^2 - (\alpha\lambda + \alpha - 2)ab\}}{a(b-a)^2A_{1-\alpha}} + \frac{b + A_{\alpha\lambda}}{(b-a)^3} \ln \left(\frac{a}{A_{1-\alpha}} \right), \\
\mu_{51} &= \frac{\alpha\{(1-\alpha)(b^2 + \lambda a^2) + (1 + \alpha - \lambda + \alpha\lambda)ab\}}{b(b-a)^2A_{1-\alpha}} \\
&\quad + \frac{a + A_{1-\lambda+\alpha\lambda}}{(b-a)^3} \ln \left(\frac{A_{1-\alpha}}{b} \right), \\
\mu_{52} &= \frac{\alpha(A_{\alpha-\alpha\lambda-1} - 3b)}{(b-a)^2A_{1-\alpha}} + \frac{b + A_{(1-\alpha)\lambda}}{(b-a)^3} \ln \left(\frac{b}{A_{1-\alpha}} \right), \\
\mu_{53} &= \frac{(\lambda - \alpha - \alpha\lambda)(a + A_{1-\alpha})}{(b-a)^2A_{1-\alpha}} + \frac{a + A_{(1-\alpha)\lambda}}{(b-a)^3} \ln \left(\frac{A_{1-\lambda+\alpha\lambda}}{A_{1-\alpha}} \right),
\end{aligned}$$

$$\begin{aligned}
\mu_{54} &= \frac{(1-\alpha)\lambda(a+b)}{b(b-a)^2} + \frac{a+A_{1-\lambda+\alpha\lambda}}{(b-a)^3} \ln\left(\frac{A_{1-\lambda+\alpha\lambda}}{b}\right), \\
\mu_{55} &= \frac{(\alpha-\lambda+\alpha\lambda)(b+A_{1-\alpha})}{(b-a)^2(\alpha a+(1-\alpha)b)} + \frac{b+A_{1-\lambda+\alpha\lambda}}{(b-a)^3} \ln\left(\frac{A_{1-\alpha}}{A_{1-\lambda+\alpha\lambda}}\right), \\
\mu_{56} &= \frac{2(\alpha-1)\lambda}{(b-a)^2} + \frac{b+A_{1-\lambda+\alpha\lambda}}{(b-a)^3} \ln\left(\frac{b}{A_{1-\lambda+\alpha\lambda}}\right), \\
\mu_{61} &= \frac{\alpha^{1+p}\lambda^{1+p}}{(1+p)a^{2p}} {}_2F_1[1, 2p, 2+p, -\frac{\alpha\lambda(b-a)}{a}], \\
\mu_{62} &= \frac{A_{\alpha\lambda}^p}{(a-b)^{1+p}} \left\{ A_{\alpha\lambda}^{1-2p} \beta[1-2p, 1+p] \right. \\
&\quad \left. + \frac{A_{1-\alpha}^{1-2p}}{2p-1} {}_2F_1[1-2p, -p, 2-2p, \frac{A_{1-\alpha}}{A_{\alpha\lambda}}] \right\}, \\
\mu_{63} &= \frac{A_{\alpha\lambda}^p}{(2p-1)(b-a)^{1+p}} \left\{ a^{1-2p} {}_2F_1[1-2p, -p, 2-2p, \frac{a}{A_{\alpha\lambda}}] \right. \\
&\quad \left. - A_{1-\alpha}^{1-2p} {}_2F_1[1-2p, -p, 2-2p, \frac{A_{1-\alpha}}{A_{\alpha\lambda}}] \right\}, \\
\mu_{64} &= \frac{A_{1-\lambda+\alpha\lambda}^p}{(2p-1)(a-b)^{1+p}} \left\{ b^{1-2p} {}_2F_1[1-2p, -p, 2-2p, \frac{b}{A_{1-\lambda+\alpha\lambda}}] \right. \\
&\quad \left. - A_{1-\alpha}^{1-2p} {}_2F_1[1-2p, -p, 2-2p, \frac{A_{1-\alpha}}{A_{1-\lambda+\alpha\lambda}}] \right\}, \\
\mu_{65} &= \frac{A_{1-\lambda+\alpha\lambda}^p}{(b-a)^{1+p}} \left\{ A_{1-\lambda+\alpha\lambda}^{1-2p} \beta[1-2p, 1+p] \right. \\
&\quad \left. + \frac{A_{1-\alpha}^{1-2p}}{2p-1} {}_2F_1[1-2p, -p, 2-2p, \frac{A_{1-\alpha}}{A_{1-\lambda+\alpha\lambda}}] \right\}, \\
\mu_{66} &= \frac{A_{1-\lambda+\alpha\lambda}^p}{(a-b)^{1+p}} \left\{ A_{1-\lambda+\alpha\lambda}^{1-2p} \beta[1-2p, 1+p] \right. \\
&\quad \left. + \frac{b^{1-2p}}{2p-1} {}_2F_1[1-2p, -p, 2-2p, \frac{b}{A_{1-\lambda+\alpha\lambda}}] \right\}, \\
\nu_{71} &= -\frac{3a+b}{2(b-a)(b^2-a^2)} + \frac{a+b}{(b-a)^3} \ln\left(\frac{2b}{a+b}\right), \\
\nu_{72} &= \frac{a+3b}{2(b-a)(b^2-a^2)} - \frac{2b}{(b-a)^3} \ln\left(\frac{2b}{a+b}\right), \\
\nu_{73} &= \frac{3a+b}{2(b-a)(b^2-a^2)} - \frac{2a}{(b-a)^3} \ln\left(\frac{a+b}{2a}\right), \\
\nu_{74} &= -\frac{a+3b}{2(b-a)(b^2-a^2)} + \frac{a+b}{(b-a)^3} \ln\left(\frac{a+b}{2a}\right),
\end{aligned}$$

$$\begin{aligned}\nu_{75} &= -\frac{1}{b^2 - a^2} + \frac{1}{(b-a)^2} \ln \left(\frac{a+b}{2a} \right), \\ \nu_{76} &= \frac{1}{b^2 - a^2} - \frac{1}{(b-a)^2} \ln \left(\frac{2b}{a+b} \right), \\ \nu_{77} &= -\frac{a}{b(b^2 - a^2)} - \frac{1}{(b-a)^2} \ln \left(\frac{2b}{a+b} \right), \\ \nu_{78} &= \frac{b}{a(b^2 - a^2)} - \frac{1}{(b-a)^2} \ln \left(\frac{2a+b}{2a} \right).\end{aligned}$$

In order to find some new inequalities of Hermite-Hadamard-like and Simpson-like type inequalities connected with the left and right parts of (3) for functions whose derivatives are harmonically convex, we need the following lemma:

Lemma 1. *Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I such that $f \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. Then the following identity holds:*

$$\begin{aligned}I_f(\lambda, \alpha, a, b) &\equiv \lambda \left\{ (1-\alpha)f(a) + \alpha f(b) \right\} + (1-\lambda)f\left(\frac{ab}{A_{1-\alpha}}\right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ &= ab(a-b) \left[\int_0^{1-\alpha} \frac{t-\alpha\lambda}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \right. \\ &\quad \left. + \int_{1-\alpha}^1 \frac{t-1+\lambda(1-\alpha)}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \right] \quad (6)\end{aligned}$$

for $t \in [0, 1]$, where $A_t = (1-t)a + tb$.

Proof By the simple calculation, this is proved.

In Lemma 1, if we choose $\lambda = 0, \frac{1}{3}, 1$ and $\alpha = \frac{1}{2}$, then we get

$$\begin{aligned}(a) \quad &\frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx - f\left(\frac{2ab}{a+b}\right) \\ &= ab(b-a) \left[\int_0^{\frac{1}{2}} \frac{t}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt + \int_{\frac{1}{2}}^1 \frac{t-1}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \right], \\ (b) \quad &\frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \\ &= ab(b-a) \int_0^1 \left(\frac{1}{2} - t \right) \frac{1}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt,\end{aligned}$$

$$(c) \quad \begin{aligned} & \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx - \frac{1}{6} \left\{ f(a) + 4f\left(\frac{2ab}{a+b}\right) + f(b) \right\} \\ &= ab(b-a) \left[\int_0^{\frac{1}{2}} \frac{t-\frac{1}{6}}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt + \int_{\frac{1}{2}}^1 \frac{t-\frac{5}{6}}{A_t^2} f'\left(\frac{ab}{A_t}\right) dt \right]. \end{aligned}$$

Now we turn our attention to establish inequalities of Hermit-Hadamard-like and Simpson-like type for differentiable harmonically convex functions.

Theorem 2.1. *Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

(a) *If $\alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha)$, then we have*

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \left[(\mu_{11} + \mu_{12})^{\frac{1}{p}} \left\{ (\mu_{31} + \mu_{32}) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (\mu_{33} + \mu_{34}) |f'(b)|^q \right\}^{\frac{1}{q}} + (\mu_{22} + \mu_{23})^{\frac{1}{p}} \right. \\ & \quad \times \left. \left\{ (\mu_{53} + \mu_{54}) |f'(a)|^q + (\mu_{55} + \mu_{56}) |f'(b)|^q \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

(b) *If $\alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha$, then we have*

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \left[(\mu_{11} + \mu_{12})^{\frac{1}{p}} \left\{ (\mu_{31} + \mu_{32}) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (\mu_{33} + \mu_{34}) |f'(b)|^q \right\}^{\frac{1}{q}} + \mu_{21}^{\frac{1}{p}} \right. \\ & \quad \times \left. \left\{ (\mu_{51} + \mu_{52}) |f'(a)|^q \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

(c) *If $1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha)$, then we have*

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \left[\mu_{13}^{\frac{1}{p}} \left\{ \mu_{41} |f'(a)|^q + \mu_{42} |f'(b)|^q \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\mu_{22} + \mu_{23})^{\frac{1}{p}} \left\{ (\mu_{53} + \mu_{54}) |f'(a)|^q \right. \right. \\ & \quad \left. \left. + (\mu_{55} + \mu_{56}) |f'(b)|^q \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Proof From Lemma 1 and by the power mean integral inequality, we have

$$\begin{aligned}
& \left| I_f(\lambda, \alpha, a, b) \right| \\
&= \left| \lambda \left\{ (1 - \alpha)f(a) + \alpha f(b) \right\} \right. \\
&\quad \left. + (1 - \lambda)f\left(\frac{ab}{A_{1-\alpha}}\right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
&\leq ab(b-a) \left[\int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \right. \\
&\quad \left. + \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \right] \\
&\leq ab(b-a) \left[\left(\int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} dt \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left(\int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} dt \right)^{1-\frac{1}{q}} \\
&\quad \times \left. \left(\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{\frac{1}{q}} \right]. \tag{7}
\end{aligned}$$

Note that:

(a) (i) If $\alpha\lambda \leq 1 - \alpha$, then we have

$$\begin{aligned}
& \int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} dt \\
&= \int_0^{\alpha\lambda} \frac{\alpha\lambda - t}{A_t^2} dt + \int_{\alpha\lambda}^{1-\alpha} \frac{t - \alpha\lambda}{A_t^2} dt \\
&= \mu_{11} + \mu_{12}.
\end{aligned}$$

(ii) If $\alpha\lambda \geq 1 - \alpha$, then we have

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} dt = \int_0^{1-\alpha\lambda} \frac{\alpha\lambda - t}{A_t^2} dt = \mu_{13}.$$

(b) (i) If $1 - \lambda(1 - \alpha) \leq 1 - \alpha$, i.e., $\alpha \leq \lambda(1 - \alpha)$, then we have

$$\begin{aligned}
& \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} dt \\
&= \int_{1-\alpha}^1 \frac{t - 1 + \lambda(1 - \alpha)}{A_t^2} dt = \mu_{21}.
\end{aligned}$$

(ii) If $1 - \lambda(1 - \alpha) \geq 1 - \alpha$, i.e., $\alpha \geq \lambda(1 - \alpha)$, then we have

$$\begin{aligned} & \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} dt \\ &= \int_{1-\alpha}^{1-\lambda(1-\alpha)} \frac{1 - \lambda(1 - \alpha) - t}{A_t^2} dt \\ &+ \int_{1-\lambda(1-\alpha)}^1 \frac{t - 1 + \lambda(1 - \alpha)}{A_t^2} dt \\ &= \mu_{22} + \mu_{23}. \end{aligned}$$

Since $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$, we know that for $t \in [0, 1]$

$$\left| f' \left(\frac{ab}{tb + (1-t)a} \right) \right|^q \leq t |f'(a)|^q + (1-t) |f'(b)|^q,$$

hence, by simple calculation, we have that:

(c) (i) If $\alpha\lambda \leq 1 - \alpha$, then we have

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \\ & \leq \int_0^{\alpha\lambda} \frac{\alpha\lambda - t}{A_t^2} \left\{ t |f'(a)|^q + (1-t) |f'(b)|^q \right\} dt \\ &+ \int_{\alpha\lambda}^{1-\alpha} \frac{t - \alpha\lambda}{A_t^2} \left\{ t |f'(a)|^q + (1-t) |f'(b)|^q \right\} dt \\ &= \{\mu_{31} + \mu_{32}\} |f'(a)|^q + \{\mu_{33} + \mu_{34}\} |f'(b)|^q. \end{aligned}$$

(ii) If $\alpha\lambda \geq 1 - \alpha$, then we have

$$\begin{aligned} & \int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \\ & \leq \int_0^{1-\alpha\lambda} \frac{\alpha\lambda - t}{A_t^2} \left\{ t |f'(a)|^q + (1-t) |f'(b)|^q \right\} dt \\ &= \left\{ \int_0^{1-\alpha} \frac{(\alpha\lambda - t)t}{A_t^2} dt \right\} |f'(a)|^q \\ &+ \left\{ \int_0^{1-\alpha} \frac{(\alpha\lambda - t)(1-t)}{A_t^2} dt \right\} |f'(b)|^q \\ &= \mu_{41} |f'(a)|^q + \mu_{42} |f'(b)|^q. \end{aligned}$$

(d) (i) If $1 - \lambda(1 - \alpha) \leq 1 - \alpha$, i.e., $\alpha \leq \lambda(1 - \alpha)$, then we have

$$\begin{aligned} & \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \\ & \leq \int_{1-\alpha}^1 \left\{ \frac{t - 1 + \lambda(1 - \alpha)}{A_t^2} \right\} \left\{ t |f'(a)|^q + (1 - t) |f'(b)|^q \right\} dt \\ & = \mu_{51} |f'(a)|^q + \mu_{52} |f'(b)|^q. \end{aligned}$$

(ii) If $\alpha\lambda \geq 1 - \alpha$, then we have

$$\begin{aligned} & \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \\ & \leq \int_{1-\alpha\lambda}^{1-\lambda(1-\alpha)} \left\{ \frac{1 - \lambda(1 - \alpha) - t}{A_t^2} \right\} \left\{ t |f'(a)|^q + (1 - t) |f'(b)|^q \right\} dt \\ & \quad + \int_{1-\lambda(1-\alpha)}^1 \left\{ \frac{t - 1 + \lambda(1 - \alpha)}{A_t^2} \right\} \left\{ t |f'(a)|^q + (1 - t) |f'(b)|^q \right\} dt \\ & = \left\{ \int_{1-\alpha}^{1-\lambda(1-\alpha)} \frac{(1 - \lambda(1 - \alpha) - t)t}{A_t^2} dt \right. \\ & \quad \left. + \int_{1-\lambda(1-\alpha)}^1 \frac{(t - 1 + \lambda(1 - \alpha))t}{A_t^2} dt \right\} |f'(a)|^q \\ & \quad + \left\{ \int_{1-\alpha}^{1-\lambda(1-\alpha)} \frac{(1 - \lambda(1 - \alpha) - t)(1 - t)}{A_t^2} dt \right. \\ & \quad \left. + \int_{1-\lambda(1-\alpha)}^{1-\alpha} \frac{(t - 1 + \lambda(1 - \alpha))(1 - t)}{A_t^2} dt \right\} |f'(a)|^q \\ & = \{\mu_{53} + \mu_{54}\} |f'(a)|^q + \{\mu_{55} + \mu_{56}\} |f'(a)|^q. \end{aligned}$$

By substituting (a)-(d) in (7), we get the desired result.

Corollary 2.1. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx - f \left(\frac{2ab}{a+b} \right) \right| \\ & \leq ab(b-a) \left[\nu_{75}^{\frac{1}{p}} \left\{ \nu_{71} |f'(a)|^q + \nu_{72} |f'(b)|^q \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \nu_{76}^{\frac{1}{p}} \left\{ \nu_{73} |f'(a)|^q + \nu_{74} |f'(b)|^q \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Theorem 2.2. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$, then the following inequality holds:

(a) If $\alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha)$, then we have

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \\ & \quad \times \left[(\mu_{61} + \mu_{62})^{\frac{1}{p}} \left\{ \frac{\alpha a \{ |f'(A_{1-\alpha})|^q + |f'(b)|^q \}}{2A_{1-\alpha}} \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\mu_{65} + \mu_{66})^{\frac{1}{p}} \left\{ \frac{(1-\alpha)b \{ |f'(a)|^q + |f'(A_{1-\alpha})|^q \}}{2A_{1-\alpha}} \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

(b) If $\alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha$, then we have

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \\ & \quad \times \left[(\mu_{61} + \mu_{62})^{\frac{1}{p}} \left\{ \frac{\alpha a \{ |f'(A_{1-\alpha})|^q + |f'(b)|^q \}}{2A_{1-\alpha}} \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \mu_{64}^{\frac{1}{p}} \left\{ \frac{(1-\alpha)b \{ |f'(a)|^q + |f'(A_{1-\alpha})|^q \}}{2A_{1-\alpha}} \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

(c) If $1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha)$, then we have

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \left[\mu_{63}^{\frac{1}{p}} \left\{ \frac{\alpha a \{ |f'(A_{1-\alpha})|^q + |f'(b)|^q \}}{2A_{1-\alpha}} \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + (\mu_{65} + \mu_{66})^{\frac{1}{p}} \left\{ \frac{(1-\alpha)b \{ |f'(a)|^q + |f'(A_{1-\alpha})|^q \}}{2A_{1-\alpha}} \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Proof From Lemma 1 and by the Hölder integral inequality, we have

$$\begin{aligned} & |I_f(\lambda, \alpha, a, b)| \\ & \leq ab(b-a) \left[\int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \right. \\ & \quad \left. + \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \right] \\ & \leq ab(b-a) \left[\left(\int_0^{1-\alpha} \frac{|t - \alpha\lambda|^p}{A_t^{2p}} dt \right)^{\frac{1}{p}} \left(\int_0^{1-\alpha} \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|^p}{A_t^{2p}} dt \right)^{\frac{1}{p}} \left(\int_{1-\alpha}^1 \left| f'\left(\frac{ab}{A_t}\right) \right|^q dt \right)^{\frac{1}{q}} \right]. \end{aligned}$$

$$+ \left(\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|^p}{A_t^{2p}} dt \right)^{\frac{1}{p}} \left(\int_{1-\alpha}^1 \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt \right)^{\frac{1}{q}}. \quad (8)$$

Note that:

(a) (i) If $\alpha\lambda \leq 1 - \alpha$, then we have

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda|^p}{A_t^{2p}} dt = \mu_{61} + \mu_{62}.$$

(ii) If $\alpha\lambda \geq 1 - \alpha$, then we have

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda|^p}{A_t^{2p}} dt = \mu_{63}.$$

(b) (i) If $1 - \lambda(1 - \alpha) \leq 1 - \alpha$, i.e., $\alpha \leq \lambda(1 - \alpha)$, then we have

$$\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|^p}{A_t^{2p}} dt = \mu_{64}.$$

(ii) If $1 - \lambda(1 - \alpha) \geq 1 - \alpha$, i.e., $\alpha \geq \lambda(1 - \alpha)$, then we have

$$\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|^p}{A_t^{2p}} dt = \mu_{65} + \mu_{66}.$$

(c) By Theorem 1.1, we have that

$$\begin{aligned} \int_0^{1-\alpha} \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt &= \frac{ab}{b-a} \int_{A_{1-\alpha}}^b \frac{|f'(x)|^q}{x^2} dt \\ &\leq \frac{\alpha a}{A_{1-\alpha}} \left\{ \frac{|f'(A_{1-\alpha})|^q + |f'(b)|^q}{2} \right\}. \end{aligned}$$

This inequality holds for $\alpha = 0$.

(d) By Theorem 1.1, we get that

$$\begin{aligned} \int_{1-\alpha}^1 \left| f' \left(\frac{ab}{A_t} \right) \right|^q dt &= \frac{ab}{b-a} \int_a^{A_{1-\alpha}} \frac{|f'(x)|^q}{x^2} dt \\ &\leq \frac{(1-\alpha)b}{A_{1-\alpha}} \left\{ \frac{|f'(a)|^q + |f'(A_{1-\alpha})|^q}{2} \right\}. \end{aligned}$$

This inequality also holds for $\alpha = 1$.

By substituting (a)-(d) in (8), we get the desire result.

Corollary 2.2. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$

and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is harmonically convex on $[a, b]$ for $q > 1$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx - f\left(\frac{2ab}{a+b}\right) \right| \\ & \leq ab(b-a) \left(\frac{1}{2^{1+p}(1+p)} \right)^{1-\frac{1}{q}} \left[\left\{ \nu_{75} |f'(a)|^q + \nu_{76} |f'(b)|^q \right\}^{\frac{1}{q}} \right. \\ & \quad \left. + \left\{ \nu_{77} |f'(a)|^q + \nu_{78} |f'(b)|^q \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Theorem 2.3. Let $f : I \subseteq R_+ = (0, \infty) \rightarrow R$ be a differentiable function on the interior I^0 of an interval I and $f' \in L([a, b])$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|$ is harmonically convex on $[a, b]$, then the following inequality holds:

(a) If $\alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha)$, then we have

$$\begin{aligned} & \left| I_f(\lambda, \alpha, a, b) \right| \\ & \leq ab(b-a) \left[\left\{ (\mu_{31} + \mu_{32}) + (\mu_{53} + \mu_{54}) \right\} |f'(a)| \right. \\ & \quad \left. + \left\{ (\mu_{33} + \mu_{34}) + (\mu_{55} + \mu_{56}) \right\} |f'(b)| \right]. \end{aligned}$$

(b) If $\alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha$, then we have

$$\begin{aligned} & \left| I_f(\lambda, \alpha, a, b) \right| \\ & \leq ab(b-a) \left[\left\{ (\mu_{31} + \mu_{32}) + \mu_{51} \right\} |f'(a)| \right. \\ & \quad \left. + \left\{ (\mu_{33} + \mu_{34}) + \mu_{52} \right\} |f'(b)| \right]. \end{aligned}$$

(c) If $1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha)$, then we have

$$\begin{aligned} & \left| I_f(\lambda, \alpha, a, b) \right| \\ & \leq ab(b-a) \left[\left\{ \mu_{41} + (\mu_{53} + \mu_{54}) \right\} |f'(a)| \right. \\ & \quad \left. + \left\{ (\mu_{42} + (\mu_{55} + \mu_{56})) \right\} |f'(b)| \right]. \end{aligned}$$

where μ_{ij} ($i = 3, 4, 5$ and $j = 1, 2, 3, 4, 5, 6$) are defined in Theorem 2.1.

Proof From Lemma 1 and by the power mean integral inequality, we have

$$\begin{aligned} & \left| I_f(\lambda, \alpha, a, b) \right| \\ & \leq ab(b-a) \left[\int_0^{1-\alpha} \frac{|t - \alpha\lambda|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \right. \\ & \quad \left. + \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)|}{A_t^2} \left| f'\left(\frac{ab}{A_t}\right) \right| dt \right]. \end{aligned}$$

Hence, by the harmonically convexity of $|f'|$ on $[a, b]$, we have

$$\begin{aligned} & \left| I_f(\lambda, \alpha, a, b) \right| \\ & \leq ab(b-a) \left[\left\{ \int_0^{1-\alpha} \frac{|t - \alpha\lambda| t}{A_t^2} dt \right. \right. \\ & \quad + \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1-\alpha)| t}{A_t^2} dt \left. \right\} |f'(a)| \\ & \quad + \left\{ \int_{1-\alpha}^1 \frac{|t - \alpha\lambda| (1-t)}{A_t^2} dt \right. \\ & \quad \left. \left. + \int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1-\alpha)| (1-t)}{A_t^2} dt \right\} |f'(b)| \right]. \end{aligned} \quad (9)$$

Note that:

(a) (i) If $\alpha\lambda \leq 1 - \alpha$, then we have

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda| t}{A_t^2} dt = \mu_{31} + \mu_{32}$$

and

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda| (1-t)}{A_t^2} dt = \mu_{33} + \mu_{34}.$$

(ii) If $\alpha\lambda \geq 1 - \alpha$, then we have

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda| t}{A_t^2} dt = \int_0^{1-\alpha\lambda} \frac{(\alpha\lambda - t)t}{A_t^2} dt = \mu_{41}$$

and

$$\int_0^{1-\alpha} \frac{|t - \alpha\lambda| (1-t)}{A_t^2} dt = \mu_{42}.$$

(b) (i) If $1 - \lambda(1 - \alpha) \leq 1 - \alpha$, i.e., $\alpha \leq \lambda(1 - \alpha)$, then we have

$$\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)| t}{A_t^2} dt = \mu_{51}$$

and

$$\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)| (1-t)}{A_t^2} dt = \mu_{52}$$

(ii) If $1 - \lambda(1 - \alpha) \geq 1 - \alpha$, i.e., $\alpha \geq \lambda(1 - \alpha)$, then we have

$$\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)| t}{A_t^2} dt = \mu_{53} + \mu_{54}$$

and

$$\int_{1-\alpha}^1 \frac{|t - 1 + \lambda(1 - \alpha)| (1 - t)}{A_t^2} dt = \mu_{55} + \mu_{56}.$$

By substituting (a)-(d) in (9), we get the desired result.

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Received: May 11, 2014