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Poincaré`s Map in a Van der Pol Equation

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Abstract

In this paper by applying a Poincaré transformation to a Van der Pol equation we obtain a new system that does not have periodic orbits.

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1 Introduction

In [1] was studied the damped Duffing equation and it was applied the Poincaré transformation. It was used a Gasull`s result [2] to study a van der Pol equation. In paper [3] was made an analysis of a generalization of a Van der Pol equation of degree five without periodic orbits in a domain on the plane. In paper [4] was made a study of a dynamical system on the plane without periodic orbits in a domain on the plane. Dulac`s criterion [5] gives sufficient conditions for the non-existence of periodic orbits of dynamical systems in simply connected regions on the plane. In this paper, we build a Dulac function for a transformed van der Pol system and prove that there aren`t any periodic orbit and also we obtain a general transformed van der Pol system without periodic orbits on the plane.

2 Preliminary Notes

Definition 2.1. *The van der Pol equation can be represented by a differential equation of the form:*

$$x'' + \epsilon(x^2 - 1)x' + x = 0$$

which can be written on the following way

$$\begin{cases} \dot{x} = y \\ \\ \dot{y} = -x - \epsilon(x^2 - 1)y \end{cases} \tag{1}$$

It is well known that the van der Pol system comes from an investigation about electrical circuits in the vacuum and the solution of this system has periodic orbits.

3 Main Results

These are the main results of the paper.

Lemma 3.1. *The system* (1) *can be transformed into* (3) *by applying Poincaré transformation* (2)*.*

Proof. Taking the following substitutions $x_1 = x$ and $x_2 = y$ and using Poincaré transformation as follows:

$$\frac{dt}{x_3^2} = d\tau,\ x_1 = \frac{1}{x_3},\ x_2 = \frac{u}{x_3}, x_3 \neq 0. \tag{2}$$

We obtain the following system

$$\begin{cases} \frac{du}{d\tau} &= -x_3^2 - \epsilon(1-x_3^2)u - u^2x_3^2 \\ \\ \frac{dx_3}{d\tau} &= -ux_3^3. \end{cases} \tag{3}$$

$\square$

Theorem 3.2. *The system* (3) *does not have periodic orbits in the domain* $0 < x_2 < 1$ *and its Dulac function is* (7)*.*

Proof. Now, we take $x_1 = u$ and $x_2 = x_3$, we obtain

$$\begin{cases} \dot{x_1} = -x_2^2 - \epsilon(1-x_2^2)x_1 - x_1^2x_2^2 \\ \\ \dot{x_2} = -x_1x_2^3. \end{cases} \tag{4}$$

So, the solution of (4) has no periodic orbits and the above system satisfies the Dulac equation

$$f_1\frac{\partial h}{\partial x_1} + f_2\frac{\partial h}{\partial x_2} = h\left[C(x_1,x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right], \tag{5}$$

where

$$\begin{aligned} f_1 &= -x_2^2 - \epsilon(1-x_2^2)x_1 - x_1^2x_2^2 \\ f_2 &= -x_1x_2^3 \\ \frac{\partial f_1}{\partial x_1} &= -\epsilon(1-x_2^2) - 2x_1x_2^2 \\ \frac{\partial f_2}{\partial x_2} &= -3x_1x_2^2. \end{aligned} \tag{6}$$

Replacing these values in (5) and supposing $\dfrac{\partial h}{\partial x_1} = 0$ and $C = -\epsilon(1-x_2^2) < 0$, we obtain:

$$\frac{\partial h}{h} = -\frac{5}{x_2}\partial x_2, \;\; x_2 \neq 0.$$

By integrating on both sides of the equation we have:

$$h = \frac{1}{x_2^5}, \;\; x_2 > 0. \tag{7}$$

$\square$

Theorem 3.3. *Let $C_1(x_1), C_2(x_2)$ be functions in $C^1(\mathbb{R})$, then the following system*

$$\begin{cases} \dot{x_1} = -\epsilon(1-x_2^2)x_1 - x_1^2x_2^2 + C_2(x_2) \\ \\ \dot{x_2} = C_1(x_1)x_2^5 - x_1x_2^3 \end{cases} \tag{8}$$

does not have periodic orbits on the domain $0 < x_2 < 1$.

Proof. From (7) we have

$$h^{-1} = x_2^5 \text{ and } \frac{\partial h}{\partial x_2} = -\frac{5}{x_2^6}.$$

Applying (5) we obtain

$$\frac{\partial f_2}{\partial x_2} - \left(\frac{5}{x_2}\right) f_2 = 2x_1x_2^2.$$

Thus, this differential equation has a solution

$$f_2 = C_1(x_1)x_2^5 - x_1x_2^3. \tag{9}$$

Now, from (6) and integrating on both sides of this equation we have:

$$f_1 = -\epsilon(1-x_2^2)x_1 - x_1^2x_2^2 + C_2(x_2). \tag{10}$$

Then, from (9) and (10) we obtain the system (8). □

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