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Antibound State for Klein-Gordon Equation

Ana-Magnolia Marin-Ramirez

Faculty of Exact and Natural Sciences, University of Cartagena
Campus San Pablo, Avenue of Consulado
Cartagena de Indias, Bolivar, Colombia

Ruben-Dario Ortiz-Ortiz

Faculty of Exact and Natural Sciences, University of Cartagena
Campus San Pablo, Avenue of Consulado
Cartagena de Indias, Bolivar, Colombia

Randy Zabaleta-Mesino

Faculty of Exact and Natural Sciences, University of Cartagena
Campus San Pablo, Avenue of Consulado
Cartagena de Indias, Bolivar, Colombia

Abstract

In this paper we construct an asymptotic for resonance of a wave function associated with the Klein-Gordon equation in presence of a potential barrier. To achieve this, we reduce the main differential equation to an integral equation using Green's function, Fourier transform and Neumann series.

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1 Introduction

It was constructed an asymptotic for the Klein-Gordon equation [1].

Unbounded states were associated with the wave equation [5]. It was studied the resonance for scattered waves [4]. It was connected the Fredholm determinant approach of Froese to the Fourier transform approach of Zworski [3]. It was studied resonance for water waves [2].

Our goal is to construct an asymptotic of an unbounded solution of the Klein-Gordon equation perturbed by a potential barrier $V(x) \in C_0^\infty(\mathbb{R})$. The antibound state (it is known as a resonance) is related with a potential that satisfies $\int_{-\infty}^{\infty} V(x)dx > 0$.

2 Preliminary Notes

We study the continuous Klein-Gordon equation

$$\Psi_{tt} - \triangle\Psi + m^2\Psi + \epsilon V(x)\Psi = 0, \quad m > 0, \quad \epsilon \to 0.$$

Now, we are looking for the solution that satisfies the definition (2.1) in the form $\Psi = e^{iwt}\varphi(x)$, where w is the frequency, we obtain

$$-\varphi_{xx}(x) + m^2\varphi(x) + \epsilon V(x)\varphi(x) = \lambda\varphi(x), \quad \lambda = w^2, \tag{1}$$

where $V(x) = 0$ for $x > r$ and $x < -r$ with r sufficiently large. The continuous spectrum of equation (1) coincides with the continuous spectrum of the unperturbed equation when $\epsilon = 0$ and it is given by $[m^2, \infty)$.

Definition 2.1. *A solution $\varphi(x)$ of equation (1) is called a resonance if φ satisfies*

$$\varphi(x) \propto e^{\beta|x|} \qquad |x| \to \infty \tag{2}$$

with $\beta > 0$ and $\lambda = m^2 - \beta^2$.

3 Main Result

The main result is as follows

Theorem 3.1. *Let $\int_{-\infty}^{\infty} V(x)dx > 0$. Then for ϵ sufficiently small, the equation (1) has a resonance for $\lambda = m^2 - \beta^2$, where*

$$\beta = \frac{\epsilon}{2}\widetilde{V}(0) + O(\epsilon^2). \tag{3}$$

Proof. We consider the problem (1) and taking $\lambda = m^2 - \beta^2$ we obtain

$$-\varphi_{xx}(x) + \beta^2\varphi(x) = -\epsilon V(x)\varphi(x).$$

We define $L = -\frac{d^2}{dx^2} + \beta^2$ and $g(x) = -\epsilon V(x)\varphi(x)$. Using Green's functions $L(G(x,\xi)) = \delta(x-\xi)$, where $G(x-\xi) = \frac{1}{2\beta}e^{-\beta|x-\xi|}$ [1]. Thus, the solution for $L\varphi = g$ is given by $\varphi = G * g$, where g has compact support. Now, we are looking for the solution of problem (1)

$$\varphi(x) = [G * (-\epsilon V\varphi)](x),$$

applying Fourier transform

$$\widetilde{\varphi}(p) = \frac{-\epsilon}{p^2+\beta^2}\widetilde{V\varphi}(p), \tag{4}$$

where $\widetilde{G}(p) = \frac{1}{p^2+\beta^2}$

$$(p^2+\beta^2)\widetilde{\varphi}(p) = \widetilde{A}(p). \tag{5}$$

We know that outside the support of $V(x)$, $|x| > r$, $\varphi(x) = A_1e^{-\beta x} + A_2e^{\beta x}$. So the sought solution is of the form

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\frac{\widetilde{A}(p)}{p^2+\beta^2}dp + A_1e^{-\beta x} + A_2e^{\beta x}, \tag{6}$$

for $|x| \to \infty$. To study the behavior of (6), we define the following contours around the simple poles

$$D_+ = \{|x| \geq 1, y = 0\} \cup \{x+iy : x^2+y^2 = 1, y > 0\}$$

$$D_- = \{|x| \geq 1, y = 0\} \cup \{x+iy : x^2+y^2 = 1, y < 0\}.$$

Applying the Cauchy residue theorem to (6), we have

$$\varphi(x) = \frac{1}{2\pi}\int_{D_+} e^{ipx}\frac{\widetilde{A}(p)}{p^2+\beta^2}dp + \left(\frac{\widetilde{A}(i\beta)}{2\beta} + A_1\right)e^{-\beta x} + A_2e^{\beta x}, \tag{7}$$

for $x > 0$. Considering the right hand side of (7) and $A_1 = -\frac{\widetilde{A}(i\beta)}{2\beta}$, we have

$$\varphi(x) = A_2e^{\beta x} + \frac{1}{2\pi}e^{-x}\int_{D_+-\{i\}}\frac{e^{ip}\widetilde{A}(p+i)}{(p+i)^2+\beta^2}dp.$$

So, $\varphi(x) = A_2e^{\beta x} + O(e^{-x})$, when $x \to \infty$.

Analogously, $\varphi(x) = A_1 e^{-\beta x} + O(e^x)$, when $x \to -\infty$ and $A_2 = -\frac{\widetilde{A}(-i\beta)}{2\beta}$.

Therefore,

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\frac{\widetilde{A}(p)}{p^2+\beta^2}dp - \frac{\widetilde{A}(i\beta)}{2\beta}e^{-\beta x} - \frac{\widetilde{A}(-i\beta)}{2\beta}e^{\beta x}. \quad (8)$$

The Fourier transform of (8) has the form

$$\widetilde{\varphi}(p) = \frac{\widetilde{A}(p)}{p^2+\beta^2} + 2\pi A_1\delta(p-i\beta) + 2\pi A_2\delta(p+i\beta). \quad (9)$$

Substituting (9) into (4), we obtain

$$\widetilde{A}(p) = -\frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\widetilde{V}(p-\xi)\frac{\widetilde{A}(\xi)}{\xi^2+\beta^2}d\xi - \epsilon A_1\widetilde{V}(p-i\beta) - \epsilon A_2\widetilde{V}(p+i\beta). \quad (10)$$

Applying the Cauchy residue theorem to the equation (10), we obtain

$$\widetilde{A}(p) = -\frac{\epsilon}{2\pi}\int_{D_+}\widetilde{V}(p-\xi)\frac{\widetilde{A}(\xi)}{\xi^2+\beta^2}d\xi + \epsilon\left(\frac{A(-i\beta)}{2\beta}\right)\widetilde{V}(p+i\beta). \quad (11)$$

We define Ω as the set of bounded analytic functions on $B_1 = \{z \in C, |\Im_z| < 1\}$, with the norm $\| \varphi \| = sup_{z\in B_1}|\varphi(z)|$, and the operator $T_\beta : \Omega \to \Omega$ by

$$[T_\beta\widetilde{A}(\xi)](p) = \int_{D_+}\widetilde{V}(p-\xi)\frac{\widetilde{A}(\xi)}{\xi^2+\beta^2}d\xi, \quad p \in \Omega. \quad (12)$$

We can rewrite the equation (11)

$$[(1+\epsilon T_\beta)\widetilde{A}(\xi)](p) = \epsilon\left(\frac{\widetilde{A}(-i\beta)}{2\beta}\right)\widetilde{V}(p+i\beta). \quad (13)$$

Since the operator T_β is bounded, therefore ϵT_β is small, it corresponds to a contraction operator then we can take its inverse, thus:

$$\widetilde{A}(p) = \epsilon\left(\frac{\widetilde{A}(-i\beta)}{2\beta}\right)[(1+\epsilon T_\beta)_{\xi\to p}]^{-1}\widetilde{V}(\xi+i\beta), \quad (14)$$

where 1 is the identity operator.
Rewriting (14) in terms of the Neumann series

$$\widetilde{A}(p) = \epsilon\left(\frac{\widetilde{A}(-i\beta)}{2\beta}\right)\sum_{n=0}^{\infty}(-1)^n\epsilon^n[T_\beta^n\widetilde{V}(\xi+i\beta)](p). \quad (15)$$

Now, let us evaluate (15) at $p = -i\beta$, we obtain

$$1 = \frac{1}{2\beta}\sum_{n=0}^{\infty}(-1)^n \epsilon^{n+1}[T_\beta^n \widetilde{V}(\xi + i\beta)](-i\beta). \tag{16}$$

Taking the main term of (16)

$$1 = \frac{\epsilon}{2\beta}\widetilde{V}(\xi + i\beta)|_{\xi=-i\beta} + O(\epsilon^2).$$

Multiplying by β, we obtain (3).

$\square$

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