Double Contraction in S -Metric Spaces

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Abstract

In this paper, we introduce double contractive mappings which are generalizations of α - ψ -contractive mappings in the context of S -metric spaces. Also, we prove existence and uniqueness of fixed points of such contractive mappings. We state some examples to demonstrate our results.

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1 Introduction

The Banach fixed point theorem [1] is a simple and powerful theorem with a wide range of applications, including iterative methods for solving linear, non-linear, differential, and integral equations. This theorem has been generalized and extended by many authors in various ways [2, 3]. Recently, Samet et al. [4] introduced the notion of α - ψ -contractive mappings and proved the related fixed point theorems. Not long ago Sedghi. et al [13] introduced the concept of S -metric spaces. Some new properties of this space was explained by [8, 9, 10]. In this paper, we combine these two notions by introducing double contractive mappings, which are a characterization of α - ψ -contractive mappings in the context of S -metric spaces. Our main results improve the existing results on the topic in the literature.

2 Basic Concepts

We briefly give some basic definitions of concepts which serve a background to this work.

Definition 2.1. Let X be a nonempty set. We call S -metric on X is a function $S: X^3 \rightarrow [0, \infty)$ which satisfies the following conditions for each $x, y, z, a \in X$

- (i) $S(x, y, z) \geq 0$,
- (ii) $S(x, y, z) = 0$ if and only if $x = y = z$,
- (iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The set X with a S -metric is called S -metric space.

Example 2.1. For any metric space (X, d) , $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ is a S -metric on X .

Example 2.2. Let $\mathbb{R}$ be a real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is a S -metric on $\mathbb{R}$. This S -metric is called the usual S -metric on $\mathbb{R}$.

Lemma 2.1. (See[13]) In a S -metric space, we have $S(x, x, y) = S(y, y, x)$.

There exists a natural topology on a S -metric spaces, for more details we refer to [8].

Lemma 2.2. (See[8]) Any S -metric space is a Hausdorff space.

Lemma 2.3. Let (X, S) be a S -metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 2.4. (See[7]) Let (X, S) be a S -metric space. Then for all $x, y, z \in X$

$$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z).$$

By the following lemma, every metric space is a S -metric space.

Lemma 2.5. (See[7]) Let (X, d) be a metric space. Then we have

- (1)- $S_d(x, y, z) = d(x, z) + d(y, z)$ is a S -metric on X .
- (2)- $x_n \rightarrow x$ in (X, d) if and only if $x_n \rightarrow x$ in (X, S_d) .
- (3)- $\{x_n\}$ is Cauchy in (X, d) if and only if $\{x_n\}$ is Cauchy in (X, S_d) .
- (4)- (X, d) is complete if and only if (X, S_d) is complete.

Let Ψ be a family of functions $\psi: [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- (a1) ψ is nondecreasing,
- (a2) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that for $k \geq k_0$ and any $t \in R^+$,

$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k \quad (1)$$

These functions are known in the literature as (c) -comparison functions.

Lemma 2.6. (See [6]) *If $\psi \in \Psi$, then the following hold:*

- (b1) $(\psi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \rightarrow \infty$ for all $t \in R^+$,
- (b2) $\psi(t) < t$ for any $t \in (0, \infty)$,
- (b3) ψ is continuous at 0,
- (b4) the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in R^+$.

The following concept introduced by [4].

Definition 2.2. *Let (X, d) be a metric space and let $T : X \rightarrow X$ be a given mapping. We say that T is an α - ψ -contractive mapping if there exist two functions $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \quad (2)$$

for all $x, y \in X$.

Obviously, any mapping satisfying Banach contraction is an α - ψ -contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, for all $t \geq 0$ and some $k \in [0, +\infty)$.

Definition 2.3. *Let T be a self mapping on X and $\alpha : X \times X \rightarrow [0, \infty)$. We say that T is α -admissible if for all $x, y \in X$, we have*

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

Various examples of such mappings are presented in [4]. The main results in [4] are the following fixed point theorems.

Theorem 2.1. *Let (X, d) be a complete metric space and T be self mapping on X and α - ψ -contractive mapping. Suppose that*

- (1) T is continuous,
- (2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 0$,
- (3) T is α -admissible.

Then there exists $u \in X$ such that $Tu = u$.

Theorem 2.2. *Let (X, d) be a complete metric space and T be self mapping on X and α - ψ -contractive mapping. Suppose that*

- (1) T is α -admissible,
- (2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
- (3) if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 0$ for all n and $x_n \rightarrow x \in X$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq 1$ for all n .

Then there exists $u \in X$ such that $Tu = u$.

3 Main Result

In this part we will introduce the concept of double contractive mapping on S -metric spaces as follows:

Definition 3.1. Let (X, S) be a S -metric space and let T be a self mapping on X . We say T is a **double contractive** mapping if there exist two functions $\gamma : X^3 \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\gamma(x, x, y)S(Tx, Tx, Ty) \leq \psi(S(x, x, y)).$$

Definition 3.2. Let (X, S) be a S -metric space and let T be a self mapping on X . We say T is a **double S -contractive** mapping if there exist two functions $\gamma : X^3 \rightarrow [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$, we have

$$\gamma(x, y, Tx)S(Tx, Ty, T^2x) \leq \psi(S(x, y, T^2x)). \quad (3)$$

Definition 3.3. Let $T : X \rightarrow X$ and $\gamma : X^3 \rightarrow [0, \infty)$. We say that T is **γ -acceptable** if for all $x, y \in X$, we have

$$\gamma(x, x, y) \geq 1 \quad \Rightarrow \quad \gamma(Tx, Tx, Ty) \geq 1.$$

Example 3.1. Let $X = [0, \infty)$ and $T : X \rightarrow X$. Define $\gamma(x, x, y) : X^3 \rightarrow [0, \infty)$ by

$$Tx = \begin{cases} 2\ln x & \text{if } x \neq 0, \\ e & \text{otherwise,} \end{cases} \quad \text{and} \quad \gamma(x, x, y) = \begin{cases} e & \text{if } x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then T is γ -admissible.

Theorem 3.1. Let (X, S) be a complete S -metric space. Suppose that $T : X \rightarrow X$ is a double S -contractive mapping and satisfies the following conditions:

- (c1) T is γ -admissible,
- (c2) there exists $x_0 \in X$ such that $\gamma(x_0, x_0, Tx_0) \geq 1$,
- (c3) T is continuous.

Then there exists $u \in X$ such that $Tu = u$.

Proof. Let $x_0 \in X$ such that $\gamma(x_0, x_0, Tx_0) \geq 1$ (such a point exists from (c2)). Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_{n_0} = x_{n_0+1}$ for some n_0 , then $u = x_{n_0}$ is a fixed point of T . So, we can assume that $x_n \neq x_{n+1}$ for all n . Since T is γ -admissible, we have

$$\gamma(x_0, x_0, x_1) = \gamma(x_0, x_0, Tx_0) \geq 1 \quad \Rightarrow \quad \gamma(Tx_0, Tx_0, Tx_1) = \gamma(x_1, x_1, x_2) \geq 1.$$

Inductively, we have

$$\gamma(x_n, x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n = 0, 1, \dots \quad (4)$$

From (3) and (4), it follows that for all $n \geq 1$, we have

$$\begin{aligned}
 S(x_n, x_n, x_{n+1}) &= S(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
 &= S(Tx_{n-1}, Tx_{n-1}, T^2x_{n-1}) \\
 &\leq \gamma(x_{n-1}, x_{n-1}, Tx_{n-1})S(Tx_{n-1}, Tx_{n-1}, T^2x_{n-1}) \\
 &\leq \psi(S(x_{n-1}, x_{n-1}, Tx_{n-1})) \\
 &\leq \psi(S(x_{n-1}, x_{n-1}, x_n)).
 \end{aligned}$$

Since ψ is nondecreasing, by induction, we have

$$S(x_n, x_n, x_{n+1}) \leq \psi^n(S(x_0, x_0, x_1)) \quad \text{for all } n \geq 0. \quad (5)$$

Using (iii) and (5), we have

$$\begin{aligned}
 S(x_n, x_n, x_m) &\leq 2 \sum_{k=n}^{m-2} S(x_k, x_k, x_{k+1}) + S(x_{m-1}, x_{m-1}, x_m) \\
 &\leq 2 \sum_{k=n}^{m-2} \psi^k S(x_0, x_0, x_1) + \psi^{m-1} S(x_0, x_0, x_1)
 \end{aligned}$$

Since $\psi \in \Psi$ and $S(x_0, x_0, x_1) > 0$, by Lemma 2.6, we get

$$\lim_{n, m \rightarrow \infty} S(x_n, x_n, x_m) = 0$$

This implies that $\{x_n\}$ is a Cauchy sequence in the S -metric space (X, S) . Since (X, S) is complete, there exists $u \in X$ such that $\{x_n\}$ is convergent to u . Since T is continuous, it follows that $\{Tx_n\}$ is convergent to Tu . By the uniqueness of the limit, we get $u = Tu$, that is, u is a fixed point of T . $\square$

For next result we will omit continuity of T .

Theorem 3.2. *Let (X, S) be a complete S -metric space. Suppose that $T : X \rightarrow X$ is a double S -contractive mapping and satisfies the following conditions:*

- (d1) T is γ -admissible,
- (d2) there exists $x_0 \in X$ such that $\gamma(x_0, x_0, Tx_0) \geq 1$,
- (d3) if $\{x_n\}$ is a sequence in X such that $\gamma(x_n, x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is convergent to $x \in X$, then $\gamma(x_n, x, x_{n+1}) \geq 1$ for all n .

Then there exists $u \in X$ such that $Tu = u$.

Proof. Following the proof of Theorem (3.1), we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is a Cauchy sequence in the complete S -metric space (X, S) , that is convergent to $u \in X$. From (4) and (d3), we have

$$\gamma(x_n, x_n, u) \geq 1 \quad \text{for all } n \geq 0. \quad (6)$$

With (3) and (6), we have

$$\begin{aligned} S(x_{n+1}, Tu, x_{n+2}) &= S(Tx_n, Tu, T^2x_n) \\ &\leq \gamma(x_n, u, x_{n+1})S(Tx_n, Tu, T^2x_n) \\ &\leq \psi(S(x_n, u, x_{n+1})). \end{aligned}$$

Letting $n \rightarrow \infty$, since ψ is continuous at $t = 0$, it follows that $S(u, Tu, u) = 0$, then $u = Tu$. $\square$

Following example, show that the hypotheses in Theorems 3.1 and 3.2 do not guarantee uniqueness.

Example 3.2. Let $X = [0, \infty)$ and (X, S) be a S -metric space, where $S(x, y, z)$ be usual S -metric on X , $\forall x, y, z \in X$. Consider the self-mapping T on X given by

$$Tx = \begin{cases} 2x - \frac{7}{4} & \text{if } x > 1, \\ \frac{x}{4} & \text{if } 0 \leq x \leq 1. \end{cases}$$

Also Define $\gamma : X^3 \rightarrow [0, \infty)$ as

$$\gamma = \begin{cases} 1 & \text{if } x, y, z \in [0, 1], \\ 0 & \text{if otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for $t \geq 0$. Then we conclude that T is a double contractive mapping. In fact, for all $x, y \in X$, we have

$$\gamma(x, x, y)S(Tx, Tx, Ty) \leq S(x, x, y).$$

On the other hand, there exists $x_0 \in X$ such that $\gamma(x_0, x_0, Tx_0) \geq 1$. Indeed, for $x_0 = 1$, we have $\gamma(1, 1, T1) = \gamma(1, 1, \frac{1}{4}) = 1$. Notice also that T is continuous. To show that T satisfies all the hypotheses of Theorem 3.1, it is sufficient to observe that T is γ -admissible. For this purpose, let $x, y \in X$ such that $\gamma(x, x, y) \geq 1$, which is equivalent to saying that $x, y \in [0, 1]$. Due to the definitions of γ and T , we have

$$Tx = \frac{x}{4} \in [0, 1], \quad Ty = \frac{y}{4} \in [0, 1].$$

Hence, $\gamma(Tx, Tx, Ty) \geq 1$. As a result, all the conditions of Theorem 3.1 are satisfied. Note that Theorem 3.1 guarantees the existence of a fixed point but not the uniqueness. In this example, 0 and $\frac{7}{4}$ are two fixed points of T .

T in the following example is not continuous.

Example 3.3. Let X, S and γ be defined as in Example 3.2. Let $T : X \rightarrow X$ be a map given by

$$Tx = \begin{cases} 2x - \frac{7}{4} & \text{if } x > 1, \\ \frac{x}{3} & \text{if } 0 \leq x \leq 1. \end{cases}$$

Let $\psi(t) = \frac{t}{3}$ for $t \geq 0$. Then we conclude that T is a double contractive mapping. In fact, for all $x, y \in X$, we have

$$\gamma(x, x, y)S(Tx, Ty, Ty) \leq \frac{2}{3}G(x, y, y).$$

For uniqueness of a fixed point of T we offer following theorem.

Theorem 3.3. Adding the following condition to the hypotheses of Theorem 3.1 we obtain the uniqueness of a fixed point of T .

(c4) For all $x, y \in X$, there exists $z \in X$ such that $\gamma(x, x, z) \geq 1$ and $\gamma(y, y, z) \geq 1$.

Proof. Let $u, v \in X$ be two fixed points of T . By (c4), there exists $z \in X$ such that

$$\gamma(u, u, z) \geq 1 \quad \text{and} \quad \gamma(v, v, z) \geq 1.$$

Since T is γ -admissible, we get by induction that

$$\gamma(u, u, T^n z) \geq 1 \quad \text{and} \quad \gamma(v, v, T^n z) \geq 1 \quad \text{for all } n = 1, 2, \dots \quad (7)$$

From (7) and (3), we have

$$\begin{aligned} S(u, T^n z, u) &= S(Tu, T(T^{n-1}z), T^2u) \\ &\leq \gamma(u, T^{n-1}z, Tu)S(Tu, T(T^{n-1}z), T^2u) \\ &\leq \psi(u, T^{n-1}z, Tu)S(u, T^{n-1}z, u). \end{aligned}$$

Thus, we get by induction that

$$S(u, T^n z, u) \leq \psi^n(S(u, z, u)) \quad \text{for all } n = 1, 2, \dots$$

Letting $n \rightarrow \infty$, and since $\psi \in \Psi$, we have

$$S(u, T^n z, u) \rightarrow 0.$$

This implies that $\{T^n z\}$ is convergent to u . Similarly, we get $\{T^n z\}$ is convergent to v . By the uniqueness of the limit, we get $u = v$, that is, the fixed point of T is unique. $\square$

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