On McShane Integrable Sets and Lebesgue Measure

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Abstract

In this paper, we show that Lebesgue measurable sets are actually integrable sets. We also introduce an extended real-valued function $\mu : \mathcal{I} \rightarrow [0, +\infty]$, where $\mathcal{I}$ is the collection of all integrable subsets of $\mathbb{R}$, and show that this function is actually a measure on $\mathcal{I}$ and coincides with the Lebesgue measure.

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1 Introduction

In 1901, Henri Lebesgue introduced the concept of Lebesgue outer measure which led to the development of his own theory of integration, the Lebesgue integration. The concept of Lebesgue outer measure is a generalization of the classical notions of the length function of an interval. However, as seen in later developments in set theory, it is actually impossible to assign a “*length*”

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to all subsets of $\mathbb{R}$ in a way which preserves some natural additivity and translation invariance properties. This suggests that choosing a suitable class of measurable subsets is an essential prerequisite.

Lebesgue outer measure gave rise to a rich collection of subsets of $\mathbb{R}$, called *measurable sets*, that includes intervals. However, to determine the Lebesgue measure of a set is not an easy task for enthusiasts in Analysis.

The Henstock-Kurzweil integral is a generalization of the Riemann integral. By slightly modifying the definition of the Henstock-Kurzweil integral, Sir Edward James McShane in 1969 introduced the McShane integral which was shown to be equivalent to the Lebesgue integral. In 1996, Yang [4] have developed the theory of measure on $\mathbb{R}$ independently of Lebesgue using the well-developed theory of Henstock integration. She used the concept of *Henstock integrable sets*. However, not all Henstock integrable functions are McShane integrable. Thus, given a subset E of $\mathbb{R}$, this study utilizes a function $\mu(E)$ of E , using McShane integral, to define a measure on the collection $\mathcal{I}$ of all integrable subsets of $\mathbb{R}$ and show that McShane integrable sets and measurable sets are equivalent.

2 Preliminary Concepts and Known Results

First, we will recall some basic concepts and known results in McShane integration.

Definition 2.1 A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be *McShane integrable to a real number A on $[a, b]$* if for any $\epsilon > 0$, there exists a positive function $\delta : [a, b] \rightarrow \mathbb{R}^+$ such that for any McShane δ -fine division $D = \{([u, v], \xi)\}$ of $[a, b]$, we have

$$\left| (D) \sum f(\xi)(v - u) - A \right| < \epsilon.$$

If $f : [a, b] \rightarrow \mathbb{R}$ is McShane integrable to A on $[a, b]$, then we write $A = \int_a^b f$.

By a McShane δ -fine division $D = \{([u, v], \xi)\}$ of $[a, b]$ we mean that $[u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi))$.

In what follows, for convenience, we use a, b , etc. to denote elements in the real line $\mathbb{R}$ and α, β , etc. to denote elements in the set of extended-real numbers $\mathbb{R}^*$.

Let $[\alpha, \beta]$ be a closed interval in the set of extended-real numbers $\mathbb{R}^*$ and f be real-valued function on $[\alpha, \beta]$. If $\alpha = -\infty$ or $\beta = +\infty$, then we make the following conventions

$$f(-\infty) = f(+\infty) = 0.$$

Definition 2.2 [1] A real-valued function f is said to be **McShane integrable** to $A \in \mathbb{R}$ on $[\alpha, \beta]$ if for every $\epsilon > 0$ there exists $\delta : [\alpha, \beta] \rightarrow \mathbb{R}^+$ and a positive number Δ such that

$$\left| (D) \sum f(\xi)(v - u) - A \right| < \epsilon$$

whenever $D = \{([u, v], \xi)\}$ is a McShane δ -fine division of a closed interval $[a, b]$ with $[a, b] \supseteq [\alpha, \beta] \cap [-\Delta, \Delta]$. Also, if $f : [\alpha, \beta] \rightarrow \mathbb{R}$ is McShane integrable to A on $[\alpha, \beta]$, then we write $A = \int_{\alpha}^{\beta} f$.

Let $E \subseteq [\alpha, \beta]$. We say that $f : [\alpha, \beta] \rightarrow \mathbb{R}$ is *McShane integrable on E* if the function $f \cdot \chi_E$ is McShane integrable on $[\alpha, \beta]$ and we write

$$\int_E f = \int_{\alpha}^{\beta} (f \cdot \chi_E),$$

where χ_E the characteristic function of E .

Definition 2.3 [3] A set $E \subseteq \mathbb{R}$ is **Lebesgue measurable**, or simply **measurable**, if for any set $A \subseteq \mathbb{R}$, $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$, where m^* is the Lebesgue outer measure. Let $\mathcal{M}$ be the collection of all measurable subsets of $\mathbb{R}$. The function $m : \mathcal{M} \rightarrow [0, \infty]$ defined by $m(E) = m^*(E)$ for all $E \in \mathcal{M}$ is called the **Lebesgue measure**.

3 A Measure on $\mathcal{I}$

Definition 3.1 A set $E \subseteq \mathbb{R}$ is **integrable** if $\chi_{E \cap [a, b]}$ is McShane integrable on $[a, b]$, for all $[a, b] \subseteq \mathbb{R}$.

For each $[a, b] \in \mathbb{R}$, the functions $\chi_{\emptyset \cap [a, b]}(x) = \chi_{\emptyset}(x) = 0$ and $\chi_{\mathbb{R} \cap [a, b]}(x) = \chi_{[a, b]}(x)$, for all $x \in \mathbb{R}$, are McShane integrable on $[a, b]$. This means that $\emptyset$ and $\mathbb{R}$ are integrable sets. It can be seen also that singleton is also integrable. Henceforth, we let $\mathcal{I}$ be the collection of all integrable subsets of $\mathbb{R}$.

Now, we will look at a set-function defined on $\mathcal{I}$. This function was introduced in [2]. Define a function $\mu : \mathcal{I} \rightarrow [0, +\infty]$ by

$$\mu(E) = \int_{-\infty}^{\infty} \chi_E$$

for all $E \in \mathcal{I}$. It is worth noting that if $E \in \mathcal{I}$ is bounded, say $E \subseteq [a, b]$, then

$$\mu(E) = \int_{-\infty}^{\infty} \chi_E = \int_a^b \chi_E.$$

Clearly, $\mu(E) \geq 0$ for any $E \in \mathcal{I}$, $\mu(\emptyset) = 0$ and $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. For a bounded set $E \subseteq \mathbb{R}$ in $\mathcal{I}$, the definition of $\mu(E)$ is independent with the choice of $[a, b]$ such that $E \subseteq [a, b]$. To show this, suppose $E \in \mathcal{I}$ such that $E \subseteq [a, b]$ and $E \subseteq [c, d]$. Without loss of generality, we may assume that $a \leq c < b \leq d$. Since E is integrable, $\chi_{E \cap [a, b]}$ and $\chi_{E \cap [c, d]}$ are McShane integrable on $[a, b]$ and $[c, d]$, respectively. Since $E \subseteq [a, b]$ and $E \subseteq [c, d]$, $E \cap [a, b] = E$ and $E \cap [c, d] = E$ and

$$\int_a^b \chi_{E \cap [a, b]} = \int_a^b \chi_E \quad \text{and} \quad \int_c^d \chi_{E \cap [c, d]} = \int_c^d \chi_E.$$

Since

$$\int_a^c \chi_{E \cap [c, d]} = 0 \quad \text{and} \quad \int_b^d \chi_{E \cap [a, b]} = 0,$$

we have

$$\begin{aligned} \int_a^b \chi_E &= \int_a^c \chi_E + \int_c^b \chi_E = \int_a^c \chi_{E \cap [c, d]} + \int_c^b \chi_E = 0 + \int_c^b \chi_E \\ &= \int_b^d \chi_{E \cap [a, b]} + \int_c^b \chi_E = \int_b^d \chi_E + \int_c^b \chi_E = \int_c^d \chi_E. \end{aligned}$$

Hence for a bounded integrable set E , $\mu(E)$ is well-defined. For an unbounded integrable set E ,

$$\mu(E) = \int_{-\infty}^{\infty} \chi_E$$

may take the value ∞ . Since the McShane integral is uniquely determined, $\mu(E)$ is also well-defined for unbounded integrable set E .

Lemma 3.2 (Covering Lemma) [2] *If S is an arbitrary subset of $\mathbb{R}$ and $\delta : S \rightarrow (0, \infty)$ be a function, then there exists a countable collection of closed non-overlapping intervals $\{K_i : i = 1, 2, \dots\}$ such that for every K_i there exists $x_i \in S$ with $K_i \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))$ and $S \subseteq \bigcup_{i=1}^{\infty} K_i$.*

Lemma 3.3 *If $E \subseteq [a, b]$ and χ_E is McShane integrable on $[a, b]$, then for every $\epsilon > 0$, there exists a collection $\{I_n : n \in \mathbb{N}\}$ of open intervals such that*

$$E \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad \sum_{n=1}^{\infty} \ell(I_n) < \int_a^b \chi_E + \epsilon.$$

Proof: Let $E \subseteq [a, b] \subseteq \mathbb{R}$ and $A = \int_a^b \chi_E$. Let $\epsilon > 0$. Put $\epsilon_0 = \min\{1, \frac{\epsilon}{2(A+2)}\}$. Then there exists $\delta : [a, b] \rightarrow \mathbb{R}^+$ such that if $D = \{([u, v], \xi)\}$ is any δ -fine

division of $[a, b]$, then $0 \leq (D) \sum \chi_E(\xi)(v - u) < A + \epsilon_0$. By Lemma 3.2, there exists a countable collection $\{K_i : i \in \mathbb{N}\}$ of closed intervals such that for every i , there exists $x_i \in E$ with $K_i \subseteq (x_i - \delta(x_i), x_i + \delta(x_i))$ and $E \subseteq \bigcup_{i=1}^{\infty} K_i$. Since $E \subseteq [a, b]$ and $[a, b]$ is closed, we may assume that $K_i \subseteq [a, b]$ for all i .

Let $n \in \mathbb{N}$. Then $[a, b] \setminus \bigcup_{i=1}^n K_i$ consists of a finite number of disjoint intervals. Let $D_0 = \{([u, v], \xi)\}$ be a McShane δ -fine division of $[a, b] \setminus \bigcup_{i=1}^n K_i$. Then $D' = D_0 \cup \{(K_i, x_i) : i = 1, 2, \dots, n\}$ is a McShane δ -fine division of $[a, b]$. Hence,

$$0 \leq \sum_{D_0} \chi_E(\xi)(v - u) + \sum_{i=1}^n \chi_E(x_i) \ell(K_i) < A + \epsilon_0.$$

Thus, for all n

$$\sum_{i=1}^n \ell(K_i) \leq \sum_{D_0} \chi_E(\xi)(v - u) + \sum_{i=1}^n \chi_E(x_i) \ell(K_i) < A + \epsilon_0.$$

Since n is arbitrary, $\sum_{i=1}^{\infty} \ell(K_i) \leq A + \epsilon_0$. For each n , let I_n be an open interval with $\ell(I_n) = \ell(K_n) \cdot (1 + \epsilon_0)$ and $K_n \subseteq I_n$. Then $E \subseteq \bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$\begin{aligned} \sum_{n=1}^{\infty} \ell(I_n) &= \sum_{n=1}^{\infty} \{\ell(K_n) \cdot (1 + \epsilon_0)\} = (1 + \epsilon_0) \cdot \sum_{n=1}^{\infty} \ell(K_n) \\ &\leq (1 + \epsilon_0)(A + \epsilon_0) = A + \epsilon_0(1 + A + \epsilon_0) \\ &\leq A + \frac{\epsilon}{2(A + 2)}(1 + A + 1) = A + \frac{\epsilon}{2(A + 2)}(A + 2) = A + \frac{\epsilon}{2} \\ &< A + \epsilon. \end{aligned}$$

This proves the result. $\square$

Theorem 3.4 *If $E \subseteq [a, b]$ and χ_E is McShane integrable on $[a, b]$, then*

$$\int_a^b \chi_E = m^*(E).$$

Proof: This follows from Lemma 3.3.

Theorem 3.5 (Monotonicity) *If A and B are integrable sets such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*

Proof: For any integrable sets A and B such that $A \subseteq B$, $\chi_A(x) \leq \chi_B(x)$ for all $x \in \mathbb{R}$. Hence,

$$\mu(A) = \int_{-\infty}^{\infty} 1_A \leq \int_{-\infty}^{\infty} 1_B = \mu(B). \quad \square$$

Corollary 3.6 *If E is countable, then $\mu(E) = 0$.*

Proof: Let E be countable. Then $E = \{a_n : n \in \mathbb{N}\}$ where $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Let $E_n = \{a_n\}$ for each $n \in \mathbb{N}$. Then $\mu(E_n) = 0$ for all $n \in \mathbb{N}$ and

$$E = \bigcup_{n=1}^{\infty} E_n.$$

By Theorem 3.9, $\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$. Since $\mu(E) \geq 0$, equality holds. $\square$

Theorem 3.7 *If I is an interval, then $\mu(I) = \ell(I)$.*

Proof: If I is unbounded then $\mu(I) = \infty$. We are left to consider bounded intervals of $\mathbb{R}$. Let $[a, b] \subset \mathbb{R}$. Note that

$$\mu([a, b]) = \int_{-\infty}^{\infty} \chi_{[a, b]} = \int_{-\infty}^a \chi_{[a, b]} + \int_b^a \chi_{[a, b]} + \int_b^{\infty} \chi_{[a, b]} = \int_b^a \chi_{[a, b]} = b - a.$$

The same argument holds for the intervals $(a, b]$, $[a, b)$ and (a, b) . $\square$

Theorem 3.8 (Finitely Sub-additivity) *If $\{E_i : i = 1, 2, \dots, n\} \subseteq \mathcal{I}$, then*

$$\mu\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mu(E_i).$$

Proof: If $\mu(E_i) = \infty$ for some natural number n , then the conclusion is trivial. Suppose that $\mu(E_i) < \infty$ for every $n \in \mathbb{N}$. We proceed by induction on n . Let P_n be the given statement that

$$\mu\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mu(E_i).$$

Note that P_1 is trivial. Assume that P_n holds for $n = k$, to show that P_{k+1} is also true. Now,

$$\mu\left(\bigcup_{i=1}^k E_i\right) + \mu(E_{k+1}) \leq \sum_{i=1}^k \mu(E_i) + \mu(E_{k+1}) = \sum_{i=1}^{k+1} \mu(E_i).$$

Thus, it is sufficient to show that

$$\mu\left(\bigcup_{i=1}^{k+1} E_i\right) \leq \mu\left(\bigcup_{i=1}^k E_i\right) + \mu(E_{k+1}).$$

Note that

$$\bigcup_{i=1}^{k+1} E_i \setminus E_{k+1} \subseteq \bigcup_{i=1}^k E_i.$$

By the monotonicity of μ ,

$$\mu\left(\bigcup_{i=1}^{k+1} E_i \setminus E_{k+1}\right) \leq \mu\left(\bigcup_{i=1}^k E_i\right).$$

Since $\mu(E_{k+1}) < \infty$, we have

$$\mu\left(\bigcup_{i=1}^{k+1} E_i \setminus E_{k+1}\right) = \mu\left(\bigcup_{i=1}^{k+1} E_i\right) - \mu(E_{k+1}) \leq \mu\left(\bigcup_{i=1}^k E_i\right).$$

Finally,

$$\mu\left(\bigcup_{i=1}^{k+1} E_i\right) \leq \mu\left(\bigcup_{i=1}^k E_i\right) + \mu(E_{k+1}).$$

Thus, P_{k+1} is true. □

Theorem 3.9 (Countably Sub-additivity) *If $\{E_n : n = 1, 2, \dots\} \subseteq \mathcal{I}$ is a countable collection, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Proof: Let $\{E_n\}_{n=1}^{\infty}$ be a countable collection of integrable sets. If $\mu(E_n) = \infty$ for some natural number n , then the conclusion is trivial. Suppose that $\mu(E_n) < \infty$ for every $n \in \mathbb{N}$. Let $E = \bigcup_{i=1}^{\infty} E_i$. Then E is an integrable set and

$$\mu(E) = \int_{-\infty}^{\infty} \chi_E.$$

It is sufficient to consider the additional assumption that $E \subseteq [0, \infty)$. For every $i \in \mathbb{N}$, define $B_i = E \cap [i-1, i)$. This means that $B_i \subseteq E$ for all i , $B_i \cap B_j = \emptyset$ for every $i \neq j$ and $E = \bigcup_{i=1}^{\infty} B_i$. For each k , define

$$A_k = \bigcup_{i=1}^k B_i$$

and $f_k : \mathbb{R} \rightarrow \mathbb{R}$ by $f_k = \chi_{A_k}$. Note that $A_k \subseteq A_{k+1}$, thus, $\chi_{A_k} \leq \chi_{A_{k+1}}$. Next we will show that $f_k \rightarrow \chi_E$ pointwisely on $\mathbb{R}$. Let $x \in \mathbb{R}$. Consider the

following cases:

Case 1: $x \notin E$

This means that $x \notin B_i$ for every natural number i . Hence, $x \notin A_k$ for all k . Thus, for each k , $\chi_{A_k}(x) = 0$ and

$$\lim_{k \rightarrow \infty} \chi_{A_k}(x) = 0 = \chi_E(x).$$

Case 2: $x \in E$

This means that there exists $N \in \mathbb{N}$, such that $x \in B_N$. Hence, $x \in A_k$ for all $k \geq N$. Thus, $\chi_{A_k}(x) = 1$ for all $k \geq N$ and $\chi_E(x) = 1$. Therefore, given $\epsilon > 0$, we have

$$|f_k(x) - \chi_E(x)| = |1 - 1| = 0 < \epsilon$$

for all $k \geq N$. Hence, $\lim_{k \rightarrow \infty} f_k(x) = \chi_E(x)$ for all $x \in \mathbb{R}$. Therefore, by the Monotone Convergence Theorem, χ_E is McShane integrable and

$$\lim_{k \rightarrow \infty} \int_{-\infty}^{\infty} f_k = \int_{-\infty}^{\infty} \chi_E$$

or equivalently,

$$\lim_{k \rightarrow \infty} \mu \left(\bigcup_{i=1}^k B_i \right) = \mu(E) = \mu \left(\bigcup_{i=1}^{\infty} E_i \right). \quad (1)$$

Since $\{B_i : i \in \mathbb{N}\}$ is pairwise disjoint, by Theorem 3.8, $\mu \left(\bigcup_{i=1}^k E_i \right) = \sum_{i=1}^k \mu(E_i)$.

Hence,

$$\mu \left(\bigcup_{i=1}^{\infty} E_i \right) = \lim_{k \rightarrow \infty} \mu \left(\bigcup_{i=1}^k E_i \right) \leq \lim_{k \rightarrow \infty} \sum_{i=1}^k \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Therefore, the desired inequality is established. $\square$

Lemma 3.10 *Let E and F be integrable sets such that $E \subseteq F$ with $\mu(E) < \infty$. Then*

$$\mu(F \setminus E) = \mu(F) - \mu(E). \quad (2)$$

Proof: Note that

$$\begin{aligned} \mu(F) &= \int_{-\infty}^{\infty} \chi_F = \int_{-\infty}^{\infty} \chi_{(F \cap E) \cup (F \cap E^c)} = \int_{-\infty}^{\infty} (\chi_{F \cap E} + \chi_{F \cap E^c}) \\ &= \int_{-\infty}^{\infty} \chi_{F \cap E} + \int_{-\infty}^{\infty} \chi_{F \cap E^c} = \int_{-\infty}^{\infty} \chi_E + \int_{-\infty}^{\infty} \chi_{F \setminus E} = \mu(E) + \mu(F \setminus E). \end{aligned}$$

Since $\mu(E) < \infty$, equation (2) is satisfied. $\square$

Lemma 3.11 *If $A \subseteq \mathbb{R}$ and E_k ($k = 1, 2, \dots, n$) be integrable sets such that $E_i \cap E_j = \emptyset$ for $i \neq j$, then*

$$\mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) = \sum_{k=1}^n \mu(A \cap E_k).$$

Proof: If $\mu(A \cap E_k) = \infty$ for some $k \in \mathbb{N}$, then

$$\infty = \mu(A \cap E_k) \leq \mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right).$$

This means that $\mu(A \cap (\bigcup_{k=1}^n E_k)) = \infty$ and the equality is preserved. Thus, we are left to consider that $\mu(A \cap E_k) < \infty$ for every $k \in \mathbb{N}$. We proceed by induction on n . Let P_n be the statement that

$$\mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) = \sum_{k=1}^n \mu(A \cap E_k).$$

Note that P_1 is trivial. Assume that P_n is true. Now, from our assumption,

$$\begin{aligned} \mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) &= \sum_{k=1}^n \mu(A \cap E_k) \\ \mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) + \mu(A \cap E_{n+1}) &= \sum_{k=1}^n \mu(A \cap E_k) + \mu(A \cap E_{n+1}) \\ &= \sum_{k=1}^{n+1} \mu(A \cap E_k). \end{aligned}$$

Thus, it is sufficient to show that

$$\mu\left(A \cap \left(\bigcup_{k=1}^{n+1} E_k\right)\right) = \mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) + \mu(A \cap E_{n+1}).$$

Since $E_i \cap E_j = \emptyset$ for $i \neq j$, we have

$$\left[A \cap \left(\bigcup_{k=1}^{n+1} E_k\right)\right] \setminus (A \cap E_{n+1}) = A \cap \left(\bigcup_{k=1}^n E_k\right).$$

Also, since $\mu(A \cap E_k) < \infty$ for all $k \in \mathbb{N}$, by Lemma 3.10

$$\begin{aligned} \mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) &= \mu\left(\left[A \cap \left(\bigcup_{k=1}^{n+1} E_k\right)\right] \setminus (A \cap E_{n+1})\right) \\ &= \mu\left(A \cap \left(\bigcup_{k=1}^{n+1} E_k\right)\right) - \mu(A \cap E_{n+1}). \end{aligned}$$

Hence,

$$\mu\left(A \cap \left(\bigcup_{k=1}^{n+1} E_k\right)\right) = \mu\left(A \cap \left(\bigcup_{k=1}^n E_k\right)\right) + \mu(A \cap E_{n+1}).$$

Thus, P_{n+1} is true. Therefore, P_n is true for every $n \in \mathbb{N}$. $\square$

Theorem 3.12 (Finitely Additivity) *If $\{E_k : k = 1, 2, \dots, n\} \subseteq \mathcal{I}$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, then*

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \sum_{k=1}^n \mu(E_k).$$

Proof: In Lemma 3.11, put $A = \mathbb{R}$. Then

$$\mu\left(\bigcup_{k=1}^n E_k\right) = \mu\left(\mathbb{R} \cap \left(\bigcup_{k=1}^n E_k\right)\right) = \sum_{k=1}^n \mu(\mathbb{R} \cap E_k) = \sum_{k=1}^n \mu(E_k). \quad \square$$

Theorem 3.13 (Countably Additivity) *If $\{E_n : n = 1, 2, \dots\} \subseteq \mathcal{I}$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

Proof: By Theorem 3.9, we are left to show that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^{\infty} \mu(E_i).$$

By Theorem 3.8, we have

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i).$$

By the monotonicity of μ and the above equality, we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geq \sum_{i=1}^n \mu(E_i) \text{ for each } n \in \mathbb{N}.$$

Hence, $\sum_{i=1}^{\infty} \mu(E_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right)$. $\square$

Theorem 3.14 *The function μ is a measure on $\mathcal{I}$.*

Proof: Since $\mu(\emptyset) = 0$ and $\mu(E) \geq 0$ for all $E \in \mathcal{I}$, the conclusion follows from Theorem 3.13. $\square$

4 Integrable and Measurable Sets

Here, we give a relationship between integrable and measurable sets.

Lemma 4.1 [1] *If $E \subseteq [a, b]$ and χ_E is McShane integrable on $[a, b]$, then for all $\epsilon > 0$ there exists an open set $O \subseteq [a, b]$ such that $E \subseteq O$ and*

$$\int_a^b \chi_{O \setminus E} < \epsilon.$$

Lemma 4.2 *Let $E \subseteq [a, b]$. The following statements are equivalent:*

- (i) χ_E is McShane integrable on $[a, b]$.
- (ii) E is integrable.

Proof: (ii) $\Rightarrow$ (i) Assume that $E \subseteq [a, b]$ is integrable. Then $\chi_{E \cap [c, d]}$ is McShane integrable on $[c, d]$ for any $[c, d] \subseteq \mathbb{R}$. In particular, $\chi_E = \chi_{E \cap [a, b]}$ is McShane integrable on $[a, b]$.

(i) $\Rightarrow$ (ii) Assume that χ_E is McShane integrable on $[a, b]$. Let $[c, d] \subseteq \mathbb{R}$. Without loss of generality, we may assume that $a \leq c < b \leq d$. Then $E \cap [c, d] = E \cap [c, b]$. Note that

$$\int_c^d \chi_{E \cap [c, d]} = \int_c^d \chi_{E \cap [c, b]} = \int_c^b \chi_{E \cap [c, b]} = \int_a^b \chi_{E \cap [c, b]}.$$

Hence, it suffices to show that $\chi_{E \cap [c, b]}$ is McShane integrable on $[a, b]$. Since $[c, b] \subseteq [a, b]$ and $\chi_{E \cap [c, b]} = \chi_E \cdot \chi_{[c, b]}$, $\chi_{E \cap [c, b]}$ is McShane integrable on $[a, b]$. $\square$

Lemma 4.3 *Let $E \subseteq [a, b]$. The following statements are equivalent:*

- (i) χ_E is McShane integrable on $[a, b]$.
- (ii) E is measurable.

In this case, $m(E) = \mu(E)$, where $m(E)$ is the Lebesgue measure of E .

Proof: (i) $\Rightarrow$ (ii) Assume that χ_E is McShane integrable on $[a, b]$. Let $\epsilon > 0$. By Lemma 4.1, there exists an open set $O \subseteq [a, b]$ such that $E \subseteq O$ and

$$\int_a^b \chi_{O \setminus E} < \epsilon.$$

By Lemma 4.2, E is an integrable set. Also, since $O \subseteq [a, b]$ is open, O is also integrable. Hence, $O \setminus E \subseteq [a, b]$ is an integrable set and $\chi_{O \setminus E}$ is McShane integrable on $[a, b]$. By Theorem 3.4,

$$m^*(O \setminus E) = \int_a^b \chi_{O \setminus E} < \epsilon.$$

Hence, $m^*(O) = m^*(E \cup (O \setminus E)) \leq m^*(E) + m^*(O \setminus E) < m^*(E) + \epsilon$. Therefore, E is measurable.

(ii) $\Rightarrow$ (i) Assume that $E \subseteq [a, b]$ is measurable. Let $\epsilon > 0$. Hence, there exists an open set O_1 such that $E \subseteq O_1$ and $m(O_1) < m(E) + \epsilon$. Similarly, since $E^c = [a, b] \setminus E$ is measurable, there exists an open set O_2 such that $E^c \subseteq O_2$ and $m(O_2) < m(E^c) + \epsilon$. Since O_1 is open and $E \subseteq O_1$, for each $x \in E$ there exists $\delta_1(x) > 0$ such that $(x - \delta_1(x), x + \delta_1(x)) \subseteq O_1$. Also, since O_2 is open and $E^c \subseteq O_2$, for each $x \in E^c$ there exists $\delta_2(x) > 0$ such that $(x - \delta_2(x), x + \delta_2(x)) \subseteq O_2$. Define $\delta : [a, b] \rightarrow \mathbb{R}$ by

$$\delta(x) = \begin{cases} \delta_1(x) & , \text{ if } x \in E; \\ \delta_2(x) & , \text{ if } x \in E^c = [a, b] \setminus E. \end{cases}$$

Then $\delta(x) > 0$ for all $x \in [a, b]$. Let $D = \{([u, v], \xi)\}$ be McShane δ -fine division of $[a, b]$. For each $([u, v], \xi) \in D$ such that $\xi \in E$,

$$(u, v) \subseteq [u, v] \subseteq (\xi - \delta_1(\xi), \xi + \delta_1(\xi)) \subseteq O_1$$

implying that $\bigcup_{\xi \in E} (u, v) \subseteq O_1$. Hence,

$$\sum_{\xi \in E} (v - u) = \sum_{\xi \in E} m((u, v)) = m\left(\bigcup_{\xi \in E} (u, v)\right) \leq m(O_1) < m(E) + \epsilon.$$

Thus,

$$(D) \sum \chi_E(\xi)(v - u) - m(E) = \sum_{\xi \in E} (v - u) - m(E) < \epsilon. \quad (3)$$

Similarly, for each $([u, v], \xi) \in D$ such that $\xi \in E^c$

$$(u, v) \subseteq [u, v] \subseteq (\xi - \delta_2(\xi), \xi + \delta_2(\xi)) \subseteq O_2$$

implying that $\bigcup_{\xi \in E^c} (u, v) \subseteq O_2$. Hence,

$$\sum_{\xi \in E^c} (v - u) = \sum_{\xi \in E^c} m((u, v)) = m\left(\bigcup_{\xi \in E^c} (u, v)\right) \leq m(O_2) < m(E^c) + \epsilon.$$

Since $\chi_E = \chi_{[a, b] \setminus E^c} = \chi_{[a, b]} - \chi_{E^c}$ and $m(E) + m(E^c) = b - a$,

$$\begin{aligned} (D) \sum \chi_E(\xi)(v - u) &= (D) \sum [\chi_{[a, b]}(\xi) - \chi_{E^c}(\xi)](v - u) \\ &= (D) \sum \chi_{[a, b]}(\xi)(v - u) - (D) \sum \chi_{E^c}(\xi)(v - u) \\ &= (b - a) - (D) \sum \chi_{E^c}(\xi)(v - u) = (b - a) - \sum_{\xi \in E^c} (v - u) \\ &> (b - a) - m(E^c) - \epsilon = m(E) - \epsilon. \end{aligned}$$

Thus,

$$-\epsilon < (D) \sum \chi_E(\xi)(v - u) - m(E). \quad (4)$$

Therefore combining equations (3) and (4), $\left| (D) \sum \chi_E(\xi)(v - u) - m(E) \right| < \epsilon$. This shows that χ_E is McShane integrable on $[a, b]$ and

$$\mu(E) = \int_a^b \chi_E = m(E). \quad \square$$

Corollary 4.4 *Let $E \subseteq [a, b]$. The following statements are equivalent:*

- (i) χ_E is McShane integrable on $[a, b]$.
- (ii) E is integrable.
- (iii) E is measurable.

In this case, $m(E) = \mu(E)$, where $m(E)$ is the Lebesgue measure of E .

Proof: Follows from Lemma 4.2 and Lemma 4.3. $\square$

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