Inner Variation and the SLi -Functions

Julius V. Benitez and Recson G. Canton¹

Department of Mathematics and Statistics
College of Science and Mathematics
MSU-Iligan Institute of Technology
9200 Iligan City, Lanao del Norte, Philippines

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Abstract

In this paper, we investigate the Lusin Condition's by utilizing the concept of δ -fine. Using concepts of covering relations and inner variation, we gave an easier way of studying Lusin's condition (N). Important results regarding full and fine covering relations are presented to help build the concept of inner variation zero. Inner variation zero gives an alternative approach to measure zero independent to the theory of Lebesgue measure. We use these concepts to define SLi -functions and give some results related to absolutely continuous functions, functions having negligible variations and functions satisfying the strong Lusin conditions.

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1 Introduction

Nikolai Lusin (1883-1950) was the one who first introduced the Lusin's condition (N) [4]. Years after he died, this condition was refined and was called

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strong Lusin condition. This stronger condition is between the absolute continuity and Lusin's condition (N), and it plays an important role in many areas in mathematics, specially in the theory of integration. In particular, it was successfully used to give an alternative approach to the Henstock-Kurzweil integration in [5]. Moreover, while Lusin's condition (N) was defined using the Lebesgue measure (see [8]), the strong Lusin condition was used to define the SL -integral. It was shown that the SL -integral is equivalent to the Henstock-Kurzweil integral [10].

In [9], Thomson introduced the concept of a covering relation, which was used to define the inner variation. Chew [3], in his paper entitled “*On Henstock's Inner Variation and Strong Derivatives*”, used the concept of inner variation to discuss the strong derivatives (*McShane derivatives*). In the same paper, the Lebesgue and Bochner Integrals were characterized using strong derivatives and the Lusin condition.

2 Basic Concepts

For brevity, we use $[u, v]$ to represent a typical subinterval of $[a, b]$, namely, $[x_{i-1}, x_i]$. A non-empty finite collection $D = \{[u, v]\}$ of non-overlapping subintervals of $[a, b]$ is said to be a *partial division* of $[a, b]$ if

$$\bigcup_{[u,v] \in D} [u, v] \subseteq [a, b].$$

If equality holds, then we say that D is a *division* of $[a, b]$.

Let $[u, v]$ be a typical subinterval of $[a, b]$ and $([u, v], \xi)$ be an interval-point pair such that $\xi \in [u, v] \subseteq [a, b]$. The element ξ in $([u, v], \xi)$ is called the *tag* of $[u, v]$. By a *tagged partial division* (resp., *tagged division*) of $[a, b]$ we mean a collection $\{([u, v], \xi)\}$ of interval-point pairs such that $\{[u, v]\}$ is a partial division (resp., division) of $[a, b]$ with $\xi \in [a, b]$.

Definition 2.1 [6] Let δ be a gauge (a positive-valued function) on $[a, b]$. An interval-point pair $([u, v], \xi)$ is said to be δ -fine if

$$\xi \in [u, v] \subseteq (\xi - \delta(\xi), \xi + \delta(\xi)).$$

A tagged division $D = \{([u, v], \xi)\}$ is said to be a δ -fine if each interval-point pair in D is δ -fine.

Lemma 2.2 (Cousin's Lemma) [1] *If δ is a gauge on $[a, b]$, then there exists a δ -fine division of $[a, b]$.*

A general concept of covering relation is introduced in this section. This will be used to define inner variation zero. In what follows, $\mathcal{I}$ denotes the collection of all non-singleton closed intervals in $\mathbb{R}$.

Definition 2.3 [9] Let $E \subseteq \mathbb{R}$. A *covering relation* β on E is a subset of $\mathcal{I} \times E$ with the property that for each $x \in E$, there exists $I \in \mathcal{I}$ such that $x \in I$ and $(I, x) \in \beta$.

Definition 2.4 [9] A covering relation β is said to be *full at a point* x if there exists $\delta(x) > 0$ such that $([y, z], x) \in \beta$ whenever

$$x \in [y, z] \subseteq (x - \delta(x), x + \delta(x)).$$

Such a relation is said to be a *full covering relation on a set* E if it is full at each point of E . A covering relation β is said to be *fine at a point* x if for each $\delta(x) > 0$, there exists an interval-point pair $([y, z], x) \in \beta$ such that

$$x \in [y, z] \subseteq (x - \delta(x), x + \delta(x)).$$

β is a *fine covering relation on a set* E if it is fine at each point of E .

Ralph Henstock called a fine covering relation on E as *inner covering* of E . In this paper, we adopt this terminology.

Theorem 2.5 Let $E \subseteq \mathbb{R}$ and δ be a gauge of E . Then the collection

$$\beta = \{(I, x) \in \mathcal{I} \times E : (I, x) \text{ is } \delta\text{-fine}\}$$

is a fine covering relation on E .

We now introduce the concept of inner variation zero using full and fine covering relations. We start by defining an interval-point function.

Definition 2.6 Let β be a covering relation on E . An *interval-point function* on β is a real-valued function defined on β .

Definition 2.7 [9] Let β be a covering relation on E , and φ be an interval-point function defined on β . Define

$$Var(\varphi, \beta) = \sup \left\{ \sum_{(I, x) \in P} |\varphi(I, x)| : P \text{ is a partial division with } P \subseteq \beta \right\}.$$

$Var(\varphi, \beta)$ refers to the *variation* of φ taken relative to the covering relation β .

Definition 2.8 [9] Let $E \subseteq \mathbb{R}$, β be a covering relation on E , and φ be an interval-point function defined on β . Define the following:

$$\begin{aligned} Var^*(\varphi, E) &= \inf \left\{ Var(\varphi, \beta) : \beta \text{ is a full covering relation on } E \right\} \quad \text{and} \\ Var_*(\varphi, E) &= \inf \left\{ Var(\varphi, \beta) : \beta \text{ is a fine covering relation of } E \right\}. \end{aligned}$$

Var^* and Var_* refer to *full* and *fine variational measures*, respectively, generated by φ . Var_* is also known as the *inner variation*. In this paper, we always call Var_* the inner variation. If $\varphi(I, x) = \ell(I)$, the length of the closed interval I , then $Var_*(\varphi, E)$ is denoted by $Var_*(E)$. If $\varphi(I, x) = \ell(I)$, then the inner variation $Var_*(\varphi, E)$ is denoted by $IV(\varphi, E)$, or simply $IV(E)$. If $IV(E) = 0$, then we say that E is of *inner variation zero*. Clearly, $IV(\emptyset) = 0$.

Althroughout this paper, we use that following notations for inner variation:

$$\begin{aligned} IV(E) &= \inf \left\{ Var(\beta) : \beta \text{ is an inner covering of } E \right\}, \quad \text{where} \\ Var(\beta) &= \sup \left\{ \sum_{(I,x) \in P} \ell(I) : P \text{ is a partial division with } P \subseteq \beta \right\}. \end{aligned}$$

Theorem 2.9 [2] *A set $E \subset \mathbb{R}$ is of inner variation zero if and only if for each $\epsilon > 0$ there exists an inner covering β of E such that for each partial division $P = \{([u, v], \xi)\}$ with $P \subseteq \beta$, we have*

$$(P) \sum |v - u| < \epsilon.$$

3 Measure Zero and Inner Variation Zero

Here, we will show that the concepts of inner variation zero and measure zero are equivalent. First, we recall the meaning of measure zero.

Definition 3.1 A subset E of $\mathbb{R}$ is said to be of *measure zero*, written $m(E) = 0$, if for every $\epsilon > 0$ there exists a countable collection $\{I_n : n \in \mathbb{N}\}$ of open intervals such that $E \subseteq \bigcup_{n=1}^{+\infty} I_n$ and $\sum_{n=1}^{+\infty} \ell(I_n) < \epsilon$.

Theorem 3.2 *If $E \subseteq \mathbb{R}$ is of inner variation zero, then E is of measure zero.*

Proof: Let $\epsilon > 0$. By Theorem 2.9, we may choose an inner covering β of E such that for each $x \in E$ and each $n \in \mathbb{N}$ there exists an interval I_n satisfying

$$(I_n, x) \in \beta \quad \text{and} \quad x \in I_n \subseteq \left(x - \frac{\epsilon}{2^{n+2}}, x + \frac{\epsilon}{2^{n+2}}\right).$$

Thus, $E \subseteq \bigcup_{n=1}^{+\infty} I_n$ and

$$\sum_{n=1}^{+\infty} \ell(I_n) \leq \sum_{n=1}^{+\infty} \ell\left(\left(x - \frac{\epsilon}{2^{n+2}}, x + \frac{\epsilon}{2^{n+2}}\right)\right) = \sum_{n=1}^{+\infty} \frac{\epsilon}{2^{n+2}} \cdot 2 = \sum_{n=1}^{+\infty} \frac{\epsilon}{2^{n+1}} = \frac{\epsilon}{2} < \epsilon.$$

Therefore, E is of measure zero. $\square$

Theorem 3.3 *If $E \subseteq \mathbb{R}$ is of measure zero, then E is of inner variation zero.*

Proof: Assume $m(E) = 0$ and let $\epsilon > 0$. Then there exists a countable collection $\{I_n : n \in \mathbb{N}\}$ of open intervals such that $E \subseteq \bigcup_{n=1}^{+\infty} I_n$ and $\sum_{n=1}^{+\infty} \ell(I_n) < \epsilon$.

Note that for every $x \in E$, there exists $n \in \mathbb{N}$ such that $x \in I_n$. Since I_n is open, for every $x \in I_n$ there exists $\delta(x) > 0$ such that $(x - \delta(x), x + \delta(x)) \subseteq I_n$. This defines a gauge δ on E with the property that for each $x \in I_n$,

$$(x - \delta(x), x + \delta(x)) \subseteq I_n. \quad (1)$$

Consider $\beta = \{(I, x) : (I, x) \text{ is } \delta\text{-fine and } x \in E\}$. By Theorem 2.5, β is an inner covering of E . Let $P = \{([u_k, v_k], \xi_k)\}_{k=1}^m$ be any partial division with $P \subseteq \beta$. By (1), for every $k = 1, 2, \dots, m$ there exists $n_k \in \mathbb{N}$ such that

$$\xi_k \in [u_k, v_k] \subseteq (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k)) \subseteq I_{n_k}.$$

Since $\{[u_k, v_k] : k = 1, 2, \dots, m\}$ is non-overlapping,

$$\sum_{k=1}^m |v_k - u_k| < \sum_{n=1}^{+\infty} \ell(I_n) < \epsilon.$$

Therefore, by Theorem 2.9, E is of inner variation zero. $\square$

Corollary 3.4 *A set $E \subseteq \mathbb{R}$ is of inner variation zero if and only if E is of measure zero.*

Proof: This follows from Theorems 3.2 and 3.3. $\square$

4 SLi -Functions

Here, we define an SLi -function and give some results related to the condition (N) , negligible variation, and the strong Lusin condition. In what follows, F refers to a function $F : [a, b] \rightarrow \mathbb{R}$ and $S \subseteq [a, b]$.

Definition 4.1 A function F is said be an *SLi-function* on a set $S \subseteq [a, b]$ if for every $E \subseteq S$ with $IV(E) = 0$, $IV(F(E)) = 0$.

Example 4.2 Consider the Dirichlet function $F : [0, 1] \rightarrow \mathbb{R}$ defined by

$$F(x) = \begin{cases} 1 & , x \in \mathbb{Q} \cap [0, 1], \\ 0 & , x \in \mathbb{Q}^c \cap [0, 1]. \end{cases}$$

Since every countable subset of $\mathbb{R}$ is of inner variation zero, F is an *SLi-function* on $\mathbb{Q} \cap [0, 1]$.

By the same argument as in Example 4.2, we have the following remark:

Remark 4.3 If $F : [a, b] \rightarrow \mathbb{R}$ is a function and $S \subseteq [a, b]$ is countable, then F is an *SLi-function* on S .

Theorem 4.4 If F is an *SLi-function* on S and $S' \subseteq S$, then F an *SLi-function* on S' .

Proof: Assume that F is an *SLi-function* on S and $S' \subseteq S$. Let $E \subseteq S'$ with $IV(E) = 0$. Then $E \subseteq S$. Thus, by assumption, $IV(F(E)) = 0$. Hence, F an *SLi-function* on S' . $\square$

Definition 4.5 A function $F : [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous* on $[a, b]$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that for any collection $\{[u_k, v_k] : k = 1, 2, \dots, n\}$ of non-overlapping sub-intervals of $[a, b]$ satisfying

$$\sum_{k=1}^n |v_k - u_k| < \delta \text{ implies } \sum_{k=1}^n |F(v_k) - F(u_k)| = \sum_{k=1}^n |F(u_k, v_k)| < \epsilon.$$

Lemma 4.6 Let $E \subseteq [a, b]$ and $F : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$. If β is an inner covering of E , then $\gamma = \{(F(I), F(x)) : (I, x) \in \beta\}$ is an inner covering of $F(E)$.

Proof: Note that by the Intermediate Value Theorem, for any $[u, v] \subseteq [a, b]$, $F([u, v])$ is also a closed interval. Let $y \in F(E)$ and $\epsilon(y) > 0$. Then there exists $x \in E$ such that $y = F(x)$. By continuity of F at x , there exists $\delta(x) > 0$ such that for any $z \in [a, b]$ with $|z - x| < \delta(x)$, we have

$$|F(z) - F(x)| < \epsilon(y).$$

Since β is fine at x , there exists $(I, x) \in \beta$ such that $x \in I$ and

$$I \subseteq \left(x - \frac{\delta(x)}{2}, x + \frac{\delta(x)}{2}\right).$$

Now, for any $z \in I$, $z \in \left(x - \frac{\delta(x)}{2}, x + \frac{\delta(x)}{2}\right)$, that is, $|z - x| < \delta(x)$. Hence, $|F(z) - F(x)| < \epsilon(y)$, that is, $F(z) \in (F(x) - \epsilon(y), F(x) + \epsilon(y)) = (y - \epsilon(y), y + \epsilon(y))$. Thus, $F(I) \subseteq (y - \epsilon(y), y + \epsilon(y))$. We have shown that there exists $(F(I), y) \in \gamma$ such that $y \in F(I) \subseteq (y - \epsilon(y), y + \epsilon(y))$. Therefore, γ is fine at $y \in F(E)$. Thus, γ is an inner covering of $F(E)$. $\square$

Theorem 4.7 *Let $F : [a, b] \rightarrow \mathbb{R}$ be an increasing function. If F is absolutely continuous on $[a, b]$, then F is an SLi -function on $[a, b]$.*

Proof: Let $E \subseteq [a, b]$ with $IV(E) = 0$. Let $\eta > 0$. Since F is absolutely continuous on $[a, b]$, there exists $\delta > 0$ such that if $\{[u_k, v_k] : k = 1, 2, \dots, n\}$ of non-overlapping collection of sub-intervals of $[a, b]$ with $\sum_{k=1}^n |v_k - u_k| < \delta$, then

$$\sum_{k=1}^n |F(v_k) - F(u_k)| = \sum_{k=1}^n |F(u_k, v_k)| < \eta. \quad (2)$$

Since $IV(E) = 0$, by Theorem 2.9, there exists an inner covering β of E such that for any partial division $P = \{([u, v], x)\}$ with $P \subseteq \beta$, we have

$$(P) \sum |v - u| < \delta. \quad (3)$$

Let $\gamma = \{(F(I), F(x)) : (I, x) \in \beta\}$. By Lemma 4.6, γ is an inner covering of $F(E)$. Now, let $\pi = \{([s_k, t_k], y_k) : k = 1, 2, \dots, q\}$ be a partial division such that $\pi \subseteq \gamma$. Then for each $k = 1, 2, \dots, q$, $([s_k, t_k], y_k) \in \gamma$. By definition of γ and since F is increasing, for each $k = 1, 2, \dots, q$, there exists $([u_k, v_k], x_k) \in \beta$ with $x_k \in [u_k, v_k]$ such that $y_k = F(x_k)$, $s_k = F(u_k)$, and $t_k = F(v_k)$. Hence, $P = \{([u_k, v_k], x_k) : k = 1, 2, \dots, q\}$ is a partial division contained in β . Thus, by (3), we have

$$\sum_{k=1}^q |v_k - u_k| < \delta.$$

By (2),

$$\sum_{k=1}^q |t_k - s_k| = \sum_{k=1}^q |F(v_k) - F(u_k)| = \sum_{k=1}^q |F(u_k, v_k)| < \eta.$$

Therefore, by Theorem 2.9, $F(E)$ is of inner variation zero. $\square$

Definition 4.8 [5] A function F satisfy a *condition (N)* on S if for any set $E \subseteq S$ with $m(E) = 0$, then $m(F(E)) = 0$, where $F(E) = \{F(x) : x \in E\}$.

Example 4.9 The Dirichlet function in Example 4.2 satisfies the condition (N) on set $\mathbb{Q} \cap [0, 1]$.

By Corollary 3.4, we have the following result, which establishes the equivalence of the condition (N) and *SLi*-function.

Remark 4.10 A function $F : [a, b] \rightarrow \mathbb{R}$ is an *SLi*-function on $[a, b]$ if and only if F satisfies the condition (N) on $[a, b]$.

Definition 4.11 A function F has *negligible variation* on a set $S \subseteq [a, b]$ if for any $\epsilon > 0$ there exists a gauge δ on S such that for any δ -fine partial division $P = \{([u, v], x)\}$ of $[a, b]$ with $x \in S$, we have

$$(P) \sum |F(v) - F(u)| < \epsilon.$$

Example 4.12 The Dirichlet function in Example 4.2 does not have a negligible variation on $S = \mathbb{Q} \cap [0, 1]$. To show this, choose $\epsilon_0 = \frac{1}{2}$ and let $\delta : S \rightarrow \mathbb{R}^+$ be any gauge on S . Note that S is countable. For each $x \in S$,

$$(0, x) \cap (x - \delta(x), x) \neq \emptyset \quad \text{and} \quad (x, 1) \cap (x + \delta(x), x) \neq \emptyset.$$

Thus, there exists $u \in \mathbb{Q}$ such that $u \in (0, x) \cap (x - \delta(x), x)$ and there exists $v \in \mathbb{Q}^c$ such that $v \in (x, 1) \cap (x + \delta(x), x)$. Then $x \in [u, v] \subseteq (x - \delta(x), x + \delta(x))$. We have shown that for each $x \in S$, there exists a closed interval I with rational left endpoint and irrational right endpoint such that $x \in I$ and (I, x) is δ -fine. Now, choose any $\{r_1, r_2, \dots, r_n\} \subseteq S$. As in above, let $I_k = [u_k, v_k]$ be an interval with left and right endpoints in $\mathbb{Q}$ and $\mathbb{Q}^c$, respectively, and for all $k = 1, 2, \dots, n$, $r_k \in (r_k - \delta(r_k), r_k + \delta(r_k))$. We may assume that the intervals I_k are non-overlapping. Consider $P_0 = \{(I_k, r_k) : k = 1, 2, \dots, n\}$. Then P_0 is a δ -fine partial division of $[0, 1]$ with $r_k \in S$ for each $k = 1, 2, \dots, n$. Thus,

$$\sum_{k=1}^n |F(v_k) - F(u_k)| = \sum_{k=1}^n |1 - 0| = n \geq \frac{1}{2} = \epsilon_0.$$

Theorem 4.13 If F is absolutely continuous on $S \subseteq [a, b]$ and S is of inner variation zero, then F has negligible variation on S .

Proof: Let $\epsilon > 0$. Then there is a $\delta_0 > 0$ such that if $P = \{[u, v]\}$ is a collection of non-overlapping sub-intervals of $[a, b]$ with $(P) \sum |v - u| < \delta_0$, then

$$(P) \sum |F(v) - F(u)| < \epsilon. \tag{4}$$

Since $IV(S) = 0$, there exists an inner covering β of S such that for any partial division $P = \{([u, v], x)\}$ with $P \subseteq \beta$, we have

$$(P) \sum |v - u| < \delta_0. \tag{5}$$

Since β is fine at every $x \in S$, there exists a non-singleton closed interval I_x such that $x \in I_x$. Let $\delta_1(x) = \ell(I_x)$ for each $x \in S$. For each $x \in S$, define $\delta(x) = \min\{\delta_1(x), \delta_0\}$. Then δ is a gauge on S . Let $P = \{([u, v], x)\}$ be any δ -fine partial division of $[a, b]$ with $x \in S$. Since β is fine at every $x \in S$, $P \subseteq \beta$. Thus, by (5)

$$(P) \sum |v - u| < \delta_0.$$

By (4), $(P) \sum |F(v) - F(u)| < \epsilon$. $\square$

Theorem 4.14 *Let $F : [a, b] \rightarrow \mathbb{R}$ be an increasing continuous function. If F has negligible variation on a set $S \subseteq [a, b]$, then F is an SLi -function on S .*

Proof: Assume that $E \subseteq S$ and $IV(E) = 0$. Let $\epsilon > 0$. Then there exists a gauge δ on S such that for any δ -fine partial division $P = \{([u, v], x)\}$ of $[a, b]$ with $x \in S$, we have

$$(P) \sum |F(v) - F(u)| < \epsilon. \quad (6)$$

Let $\beta = \{([u, v], x) \in \mathcal{I}([a, b]) \times E : ([u, v], x) \text{ is } \delta\text{-fine}\}$, where $\mathcal{I}([a, b])$ is the family of all non-singleton closed sub-intervals of $[a, b]$. By Theorem 2.5, β is an inner covering of E . By Lemma 4.6, $\gamma = \{(F([u, v]), F(x)) : ([u, v], x) \in \beta\}$ is an inner covering of $F(E)$. Let $Q = \{([s_k, t_k], y_k)\}_{k=1}^q$ be any partial division with $Q \subseteq \gamma$. Then for each $k = 1, 2, \dots, q$, there exists $([u_k, v_k], x_k) \in \beta_0$ such that $y_k = F(x_k)$, $s_k = F(u_k)$, and $t_k = F(v_k)$. Let $P = \{([u_k, v_k], x_k)\}_{k=1}^q$. Then $P \subseteq \beta$, that is, P is δ -fine and $x_k \in S$ for all $k = 1, 2, \dots, q$. Hence, by (6)

$$\sum_{k=1}^q |t_k - s_k| = \sum_{k=1}^q |F(v_k) - F(u_k)| < \epsilon. \quad \square$$

Definition 4.15 [6] Let $F : [a, b] \rightarrow \mathbb{R}$ be a function, $S \subseteq [a, b]$ and m be the Lebesgue measure. A function F is said to satisfy the *strong Lusin condition* on S if for every set $E \subseteq S$ with $m(E) = 0$ and every $\epsilon > 0$, there exists a gauge δ on S such that for any δ -fine partial division $P = \{([u, v], \xi)\}$ with every $\xi \in E$, we have

$$(P) \sum |F(u, v)| < \epsilon,$$

where $F(u, v) = F(v) - F(u)$. We call such function F an *SL-function* on S .

Theorem 4.16 [10] *If F is an SL -function on $[a, b]$, then it satisfies the condition (N).*

By Theorem 4.16 and Remark 4.10, we have the following corollary:

Corollary 4.17 *If F is an SL -function on $[a, b]$, then F is an SLi -function on $[a, b]$.*

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