On the Ideal Center of a Dual Vector Lattice

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Abstract

Let X be a vector lattice with order dual X' . In this paper, we investigate when $Orth(X')$ is an ideal center in X' . In [6], Toumi answers related question posed by Wickstead in [1]. This study is the dual version of the paper of Toumi in [6].

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1 Introduction

Let X be an Archimedean vector lattice. $Orth(X)$ denotes the vector lattice of all orthomorphisms on X . $Z(X)$ denotes the sublattice of $Orth(X)$ consisting of those π for which there is non-negative real number λ with $-\lambda x \leq \pi(x) \leq \lambda x$ for all $x \in X^+$ (Positive cone of X). We say that $Z(X)$ is the ideal center of X . A vector lattice X under an associative multiplication is said to be lattice-ordered algebra (ℓ -algebra) whenever the multiplication makes X an algebra, in addition it satisfies the following property: If $x, y \in X^+$, then $xy \in X^+$, [2]. A lattice-ordered algebra X is said to be an f -algebra when $x \wedge y = 0$ implies $xz \wedge y = zx \wedge y = 0$ for each $z \in X^+$. Note that any Archimedean f -algebra is commutative. Moreover, squares in an f -algebra are positive, [3].

If X is an f -algebra, the mapping $\rho: X \rightarrow Orth(X)$ defined by $\rho(f) = \pi_f$ where $\pi_f(g) = fg$ for all $f, g \in X$ is an algebra and lattice homomorphism. If $\rho(f) \in Z(X)$, f is said to be bounded and if every element of X is bounded, the f -algebra X is said to be bounded, [5].

Let X be a vector lattice and let $0 \leq v \in X$. If for every real number $\varepsilon > 0$, there exists a natural number n_ε such that $|x_n - x| \leq \varepsilon v$ for all $n \geq n_\varepsilon$, the sequ-

ence $\{x_n\}_{n \geq 0}$ in X is called (v) relatively uniformly convergent to $x \in X$ which is denoted by $x_n \rightarrow x(v)$. That is, if $x_n \rightarrow x(v)$ for some $0 \leq v \in X$, then the sequence $\{x_n\}_{n \geq 0}$ is called (relatively) uniformly convergent to x . This is denoted by $x_n \rightarrow x(r.u)$. A vector lattice is called relatively uniformly complete if every relatively uniform Cauchy sequence in X has a unique limit. Relatively uniform limits are unique if X is Archimedean. The relatively uniform topology is denoted by $(r.u)$ topology. X is called Dedekind complete, if every non-empty order bounded set has a supremum and an infimum in X . X is called uniformly complete if $\sup\{\sum_{i=1}^n x_i : n \in N\}$ exists for every uniformly bounded sequence $(x_n)_1^\infty \subset X^+$, [7].

Let X be a topological algebra and $M(X)$ be the set of all maximal two sided ideals in X . The space $M(X)$ is equipped with the hk -topology: $S \subset M(X)$ is closed if $S = H(K(S))$, where $K(S)$ is the intersection of all ideals in S and $H(I) = \{M \in M(X) : I \subset M\}$ for any two-sided ideal I of X . If X is also equipped with a compatible topology we consider a subset $m(X)$ of $M(X)$ consisting of closed ideals. We use small letters h and k to indicate H and K restricted to $m(X)$.

A lattice norm $\|\cdot\|$ on X is called an M -norm if $\|x \vee y\| = \max\{\|x\|, \|y\|\}$ for all $x, y \in X^+$.

Note that, for unexplained terminology and notation, we refer to the standart books [3] and [7]. Next theorem is seen in the work of Basly and Triki in [5] and [6].

Theorem 1 If X is a relatively uniformly complete f -algebra, the following are equivalent:

- i. X is bounded.
- ii. Every maximal modular ring ideal in X is uniformly closed.
- iii. Every maximal modular ring ideal in X is the kernel of a lattice and algebra homomorphism $X \rightarrow R$.
- iv. Every maximal modular ring ideal in X is a maximal order ideal.

Theorem 2 [6] Consider a relatively uniformly complete unital f -algebra X . Then $M(X) = m(X)$, with respect to the relatively uniform topology, if and only if

- i. Every proper finitely generated ring ideal is contained in a uniformly closed maximal ring ideal of X .
- ii. $m(X)$ is compact in hk -topology.

Proposition 3 [6] Any maximal ring ideal of a relatively uniformly complete semiprime f -algebra X is an order ideal.

Now, we will give the dual version of the study in [6].

Theorem 4 [6] Let X be a vector lattice and X' be order dual of X . Then the following results are equivalent:

- i. $Orth(X') = Z(X')$.

- ii. $M(\text{Orth}(X')) = m(\text{Orth}(X'))$, with respect to the relatively uniform topology.
- iii. Every proper finitely generated ring ideal is contained in a uniformly closed maximal ring ideal of $\text{Orth}(X')$ and $m(\text{Orth}(X'))$ is compact in the hk -topology.

Proof: $i \Rightarrow ii$ Because X' is Dedekind complete, $\text{Orth}(X')$ is Dedekind complete. So, $\text{Orth}(X')$ is relatively uniformly complete. First, define the gauge function $P(\pi) := \inf\{\lambda \in R : -\lambda I_{X'} \leq \pi \leq \lambda I_{X'}\}$ for $\pi \in \text{Orth}(X')$. Since $\text{Orth}(X') = Z(X')$, $P(\pi)$ is an M -norm. Moreover, $(\text{Orth}(X'), P)$ is a Banach lattice. $\text{Orth}(X')$ is a unital f -algebra under composition, so $\text{Orth}(X')$ is a Banach unital f -algebra. Because every maximal ring ideal in $\text{Orth}(X')$ is uniformly closed, any maximal ring ideal in $\text{Orth}(X')$ is closed, with respect to the norm topology.

$ii \Leftrightarrow iii$ From the Theorem 2, it is proven.

$ii \Rightarrow i$ Assume by way of contradiction that $Z(X') \subsetneq \text{Orth}(X')$. There exists $\pi \in (\text{Orth}(X'))^+$ such that $\pi \notin Z(X')$. Suppose that $\varphi = \pi \wedge I_{X'} \in (\text{Orth}(X'))^+$ and $\omega_n = \pi - (\pi \wedge n\varphi)$, for all $n \in N$. Because of $\pi \notin Z(X')$, we get $\omega_n = \pi - (\pi \wedge n\varphi) > 0$. Then $\{\omega_n\}^d$ is proper uniformly closed order and ring ideal of $\text{Orth}(X')$. Moreover, we have $\{\omega_n\}^d \subset \{\omega_{n+1}\}^d$, for all $n \geq 1$. Let $A = \bigcup_{n \geq 1} \{\omega_n\}^d$, then A is a proper ring ideal of $\text{Orth}(X')$. Since $(I_{X'} - \varphi) \wedge (\pi - \varphi) = 0$, we get $\psi(I_{X'} - \varphi) \wedge (\pi - \varphi) = 0$ for all $\psi \in \text{Orth}(X')$. It follows that $\psi(I_{X'} - \varphi) = \psi - \psi\varphi \in \{\omega_1\}^d$ and therefore, $\psi - \psi\varphi \in A$ for all $\psi \in \text{Orth}(X')$. Because there exists a closed maximal ring ideal M such that $A \subset M$ and $M(\text{Orth}(X')) = m(\text{Orth}(X'))$, M is relatively uniformly closed order ideal of $\text{Orth}(X')$. So, the vector lattice quotient $\text{Orth}(X')/M$ is Archimedean.

Let $\pi_n = (\pi \vee n\varphi) - \pi$ for all $n \in N$. It follows that $\pi_n \omega_n = 0$. Therefore, $\pi_n \in \{\omega_n\}^d \subset A$. If we pass to the classes modulo M , we have $\overline{\pi_n} = \overline{(\pi \vee n\varphi) - \pi} = (\overline{\pi \vee n\varphi}) - \overline{\pi} = (\overline{\pi} \vee \overline{n\varphi}) - \overline{\pi} = \overline{0}$ for all $n \geq 1$ and so $\overline{0} \leq \overline{n\varphi} \leq \overline{\pi}$ for all $n \geq 1$. Since $\text{Orth}(X')/M$ is Archimedean, it follows that $\varphi \in M$. Since the relation $\psi - \psi\varphi \in A$, we get $\psi \in M$. Consequently we get the equation $M = \text{Orth}(X')$ which is a contradiction. This completes the proof of the theorem.

Corollary 5 [6] If X is Banach lattice then $\text{Orth}(X') = Z(X')$.

Proof: Since X is a Banach lattice, any orthomorphism π on X' is continuous, [7]. On the other hand, since the modulus of π exists [4], it shows that $\text{Orth}(X')$ is a Banach unital f -algebra under composition. It is clear that every maximal ring ideal in $\text{Orth}(X')$ is closed with respect to the norm topology, [4]. So, every maximal ring ideal in $\text{Orth}(X')$ is uniformly closed. Then $M(\text{Orth}(X')) = m(\text{Orth}(X'))$ with respect to the relatively uniform topology. From the equivalent conditions $ii \Rightarrow i$ in Theorem 4, we get $\text{Orth}(X') = Z(X')$.

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References

- [1] A. W. Wickstead, Representation and duality of multiplication operators on Archimedean Riesz spaces, *Compos. Math.*, **35**(3), (1977), 225-238.
- [2] C. B. Huijsmans and B. de Pagter, Ideal theory in f-algebras, *Transactions of the American Mathematical Society*, **269**, no. 1, (1982), 225-245.
<http://dx.doi.org/10.2307/1998601>
- [3] C. D. Aliprantis and O. Burkinshaw, Positive Operators, Springer, Dordrecht, (2006). <http://dx.doi.org/10.1007/978-1-4020-5008-4>
- [4] H. G. Dales, A. T. M. Lau, D. Strauss, Banach algebras on semigroups and on their compactifications, *Mem. Am. Math. Soc.*, **966**(i-v), (2010), 165.
<http://dx.doi.org/10.1090/s0065-9266-10-00595-8>
- [5] M. Basly, A. Triki, On uniformly closed ideals in f-algebras, In:2nd Conference, Function Spaces, Marcel Dekker, (1995), 29-33.
- [6] M. A. Toumi, When orthomorphisms are in the ideal center, *Positivity*, **18**, no.3, (2014), 579-583. <http://dx.doi.org/10.1007/s11117-013-0263-3>
- [7] P. Meyer - Nieberg, Banach Lattices, Universitext, Springer, Berlin, Germany, (1991). <http://dx.doi.org/10.1007/978-3-642-76724-1>

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