Generalized Composition Operators from Area Nevanlinna Spaces to Zygmund-type Spaces

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Abstract

The boundedness and compactness of generalized composition operators from area Nevanlinna spaces to Zygmund-type spaces and little Zygmund-type spaces are characterized.

Keywords: Generalized Composition Operator, Area Nevanlinna Spaces, Zygmund-type Spaces

1 Introduction

Let $\mathbb{C}$ be the complex plane of the open unit disk, $H(\mathbb{D})$ the space of all analytic function on $\mathbb{D}$. An function $f \in H(\mathbb{D})$ is said to belong to the Zygmund-type space $\mathcal{Z}_\mu$, if

$$\sup_{z \in \mathbb{D}} \mu(|z|) |f''(z)| < \infty$$

where μ is the normal function(see [6]). It is a Banach space with norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \mu(|z|) |f''(z)|.$$

The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ consists of those functions f in $\mathcal{Z}_\mu$ satisfying $\lim_{|z| \rightarrow 1^-} \mu(|z|)|f''(z)| = 0$ and it is easy to see that $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_\mu$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. Let $1 \leq p < \infty$ and $\alpha > -1$, an function $f \in H(\mathbb{D})$ is said to belong to the area Nevanlinna space $\mathcal{N}_\alpha^p = \mathcal{N}_\alpha^p(\mathbb{D})$, if

$$\|f\|_{\mathcal{N}_\alpha^p}^p = \int_{\mathbb{D}} [\log(1 + |f(z)|)]^p dA_\alpha(z) < \infty.$$

Let φ be an analytic self-map of $\mathbb{D}$. Then composition operator C_φ is defined by $(C_\varphi f) = f(\varphi(z))$, $f \in H(\mathbb{D})$. The composition operator has been studied by many researchers on various space. (see for example, [1, 2, 3]). In [4], the authors defined and studied the generalized composition operator

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad z \in \mathbb{D}, \quad f, g \in H(\mathbb{D})$$

and the boundedness and compactness of C_φ^g on Zygmund spaces and Bloch spaces were investigated in it. The purpose of this paper is to characterize the boundedness and compactness of the generalized composition operator C_φ^g from area Nevanlinna spaces to Zygmund-type spaces and little Zygmund-type spaces. In what follows, we use the letter C to denote a positive constant whose value may change its value at each occurrence.

2 Main results and proofs

Lemma 1. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $g \in H(\mathbb{D})$, $1 \leq p < \infty$, $\alpha > -1$. Then $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is compact if and only if for each sequence $\{f_k\}_{k \in \mathbb{N}}$ which is bounded in $\mathcal{N}_\alpha^p$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$, we have $\|C_\varphi^g f_k\|_{\mathcal{Z}_\mu} \rightarrow 0$ as $k \rightarrow \infty$.*

Since the proof of Lemma 1 is similar to Proposition 3.11 in [1], it is omitted.

Lemma 2. *A closed set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z| \rightarrow 1^-} \sup_{f \in K} \mu(|z|)|f''(z)| = 0. \quad (1)$$

The proof of it is similar to that of [5]. The next result can be found in [7, 8].

Lemma 3. *Let n be a nonnegative integer, $1 \leq p < \infty$ and $\alpha > -1$. Then there exists some C such that for each $f \in \mathcal{N}_\alpha^p$ and $z \in \mathbb{D}$,*

$$|f^{(n)}(z)| \leq \frac{1}{(1 - |z|^2)^n} \exp \left[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1 - |z|^2)^{\frac{\alpha+2}{p}}} \right]. \quad (2)$$

Theorem 4. Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \leq p < \infty$ and $\alpha > -1$. Then $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \infty \quad (3)$$

and

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \infty. \quad (4)$$

Proof. Suppose that $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded. Now taking $f(z) = z$ and $f(z) = z^2$, and obviously each of them belongs to $\mathcal{N}_\alpha^p$, and using the boundedness of the function $\varphi(z)$, we get

$$\sup_{z \in \mathbb{D}} \mu(|z|)|g'(z)| < \infty. \quad (5)$$

$$\sup_{z \in \mathbb{D}} \mu(|z|)|g(z)||\varphi'(z)| < \infty. \quad (6)$$

For $\omega \in \mathbb{D}$, set

$$\begin{aligned} f_z(\omega) = & \left[-\frac{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1 - \overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}} + 2\frac{(1 - |\varphi(z)|^2)^{\frac{\alpha+4}{p}}}{(1 - \overline{\varphi(z)}\omega)^{\frac{2\alpha+6}{p}}} \right. \\ & \left. - \frac{(1 - |\varphi(z)|^2)^{\frac{\alpha+6}{p}}}{(1 - \overline{\varphi(z)}\omega)^{\frac{2\alpha+8}{p}}} \right] \exp \left[c \frac{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1 - \overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}} \right]. \end{aligned}$$

Then $f_z \in \mathcal{N}_\alpha^p$, $f'_z(\varphi(z)) = 0$, $f''_z(\varphi(z)) = \frac{-8}{p^2} \frac{\overline{\varphi(z)}^2}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}+2}} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right]$.

Then

$$\begin{aligned} & \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}+2}} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \infty, \\ & \sup_{z \in \mathbb{D}} \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^2}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\ & \leq C \|C_\varphi^g f_z\|_{\mathcal{Z}_\mu} (1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}} < \infty. \end{aligned} \quad (7)$$

For any fixed $r \in (0, 1)$,

$$\begin{aligned} & \sup_{|\varphi(z)| > r} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\ & \leq \sup_{|\varphi(z)| > r} \frac{1}{r^2} \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^2}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \infty. \end{aligned} \quad (8)$$

$$\begin{aligned}
& \sup_{|\varphi(z)| \leq r} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\
& \leq \frac{1}{(1-r^2)^2} \sup_{|\varphi(z)| \leq r} \mu(|z|)|g(z)||\varphi'(z)| \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \infty. \quad (9)
\end{aligned}$$

Therefore, (8) and (9) yield (3). Next, set $h_z(\omega) = \frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\varphi(z)\omega)^{\frac{2\alpha+4}{p}}} \exp \left[c \frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\varphi(z)\omega)^{\frac{2\alpha+4}{p}}} \right]$.

Then $h_z \in \mathcal{N}_\alpha^p$,

$$\begin{aligned}
\infty & > \|C_\varphi^g\|_{\mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu} \|h_z\|_{\mathcal{N}_\alpha^p} \geq \|C_\varphi^g h_z\|_{\mathcal{Z}_\mu} \geq \sup_{w \in \mathbb{D}} \mu(|w|) |(C_\varphi^g h_z)''(w)| \\
& \geq C \frac{\mu(|z|)|g'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\
& \quad - C \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+2}} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{\mu(|z|)|g'(z)||\varphi(z)|}{1-|\varphi(z)|^2} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\
& \leq C \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^2 \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right]}{(1-|\varphi(z)|^2)^2} \\
& \quad + C \|C_\varphi^g h_z\|_{\mathcal{Z}_\mu} (1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}. \quad (10)
\end{aligned}$$

From (7) and (10), we get

$$\sup_{z \in \mathbb{D}} \frac{\mu(|z|)|g'(z)||\varphi(z)|}{1-|\varphi(z)|^2} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \infty. \quad (11)$$

Combining (11) with (5), similar to the former proof, we get (4).

For the converse, suppose that (3) and (4) hold. By Lemma 3, we have

$$\begin{aligned}
\mu(|z|) |(C_\varphi^g f)''(z)| & \leq \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} \exp \left[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\
& \quad + \frac{\mu(|z|)|g'(z)|}{1-|\varphi(z)|^2} \exp \left[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right].
\end{aligned}$$

$$\|C_\varphi^g f\|_{\mathcal{Z}_\mu} = |(C_\varphi^g f)(0)| + |(C_\varphi^g f)'(0)| + \sup_{z \in \mathbb{D}} \mu(|z|) |(C_\varphi^g f)''(z)| < \infty.$$

Therefore $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded. $\square$

Theorem 5. Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \leq p < \infty$ and $\alpha > -1$. Then $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is compact if and only if $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded,

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^2} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] = 0 \quad (12)$$

and

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{\mu(|z|)|g'(z)|}{1-|\varphi(z)|^2} \exp \left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] = 0. \quad (13)$$

Proof. Suppose that $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is compact. Then $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded. Let $\{z_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \rightarrow 1^-$ as $k \rightarrow \infty$. Set

$$\begin{aligned} f_k(\omega) = & \left[-\frac{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z_k)}\omega)^{\frac{2\alpha+4}{p}}} + 2\frac{(1-|\varphi(z_k)|^2)^{\frac{\alpha+4}{p}}}{(1-\overline{\varphi(z_k)}\omega)^{\frac{2\alpha+6}{p}}} \right. \\ & \left. -\frac{(1-|\varphi(z_k)|^2)^{\frac{\alpha+6}{p}}}{(1-\overline{\varphi(z_k)}\omega)^{\frac{2\alpha+8}{p}}} \right] \exp \left[\frac{c(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z_k)}\omega)^{\frac{2\alpha+4}{p}}} \right]. \end{aligned}$$

Then $f_k \in \mathcal{N}_\alpha^p$ by Theorem 4, and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. From the compactness of C_φ^g , we have $\lim_{k \rightarrow \infty} \|C_\varphi^g f_k\|_{\mathcal{Z}_\mu} = 0$. On the other hand, similar to the proof of Theorem 4, we have

$$\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^2 \exp \left[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}} \right]}{(1-|\varphi(z_k)|^2)^2} \leq C \|C_\varphi^g f_k\|_{\mathcal{Z}_\mu} (1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}. \quad (14)$$

Since $|\varphi(z_k)| \rightarrow 1^-$ as $k \rightarrow \infty$, we get

$$\lim_{|\varphi(z_k)| \rightarrow 1^-} \frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^2} \exp \left[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}} \right] = 0.$$

From which we get (12). Next, set $h_k(\omega) = \frac{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z_k)}\omega)^{\frac{2\alpha+4}{p}}} \exp \left[\frac{c(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z_k)}\omega)^{\frac{2\alpha+4}{p}}} \right]$.

Then $h_k \in \mathcal{N}_\alpha^p$ and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. From the compactness of C_φ^g , we have $\lim_{k \rightarrow \infty} \|C_\varphi^g h_k\|_{\mathcal{Z}_\mu} = 0$. On the other hand,

$$\begin{aligned} & C \|C_\varphi^g h_k\|_{\mathcal{Z}_\mu} (1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}} \\ \geq & \frac{\mu(|z_k|)|g'(z_k)||\varphi(z_k)|}{1-|\varphi(z_k)|^2} \exp \left[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}} \right] \\ & - \frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^2} \exp \left[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}} \right] \end{aligned}$$

which implies that

$$\lim_{|\varphi(z_k)| \rightarrow 1^-} \frac{\mu(|z_k|)|g'(z_k)|}{1 - |\varphi(z_k)|^2} \exp \left[\frac{c}{(1 - |\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}} \right] = 0.$$

From which we get (13).

Conversely, suppose that $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded and (12) and (13) hold. Assume $\{f_k\}_{k \in \mathbb{N}}$ is a bounded sequence in $\mathcal{N}_\alpha^p$ such that f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. By the assumption, for any $\varepsilon > 0$, there exists a $\delta \in (0, 1)$ such that

$$\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \frac{\varepsilon}{2}. \quad (15)$$

$$\frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \frac{\varepsilon}{2}. \quad (16)$$

whenever $\delta < |\varphi(z)| < 1$. By the boundedness of $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ and the proof of Theorem 4,

$$C_2 = \sup_{z \in \mathbb{D}} \mu(|z|)|g(z)||\varphi'(z)| < \infty \quad (17)$$

$$C_3 = \sup_{z \in \mathbb{D}} \mu(|z|)|g'(z)| < \infty. \quad (18)$$

Then by Lemma 3, (15), (16), (17) and (18), we have that

$$\sup_{z \in \mathbb{D}} \mu(|z|)|(C_\varphi^g f_k)''(z)| \leq C_2 \sup_{\omega \leq \delta} |f_k''(\omega)| + C_3 \sup_{\omega \leq \delta} |f_k'(\omega)| + \varepsilon.$$

Then we get

$$\|C_\varphi^g f_k\|_{\mathcal{Z}_\mu} \leq C_2 \sup_{\omega \leq \delta} |f_k''(\omega)| + C_3 \sup_{\omega \leq \delta} |f_k'(\omega)| + \varepsilon + |g(0)||f_k'(\varphi(0))|. \quad (19)$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$, Cauchy's estimation gives that f_k' and f_k'' also do as $k \rightarrow \infty$, and both $\{z \in \mathbb{D} : |z| < \delta\}$ and $\{0\}$ are compact subsets of $\mathbb{D}$. Hence, letting $k \rightarrow \infty$ in (19), we get $\lim_{k \rightarrow \infty} \|C_\varphi^g f_k\|_{\mathcal{Z}_\mu} = 0$. By Lemma 1, we see that $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is compact. $\square$

From Theorem 4 and Theorem 5, we can easily obtain the following corollary.

Corollary 6. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \leq p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0, 1)$. Then the following statements are equivalent:*

(i) $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded;

(ii) $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is compact;

(iii)

$$\sup_{z \in \mathbb{D}} \mu(|z|)|g(z)||\varphi'(z)| < \infty \text{ and } \sup_{z \in \mathbb{D}} \mu(|z|)|g'(z)| < \infty, \quad (20)$$

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] = 0, \quad (21)$$

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] = 0. \quad (22)$$

Theorem 7. Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \leq p < \infty$ and $\alpha > -1$. Then the following statements are equivalent:

(i) $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_{\mu,0}$ is bounded;

(ii) $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_{\mu,0}$ is compact;

(iii)

$$\lim_{|z| \rightarrow 1^-} \mu(|z|)|g(z)||\varphi'(z)| = 0 \text{ and } \lim_{|z| \rightarrow 1^-} \mu(|z|)|g'(z)| = 0 \quad (23)$$

$$\lim_{|z| \rightarrow 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] = 0 \quad (24)$$

and

$$\lim_{|z| \rightarrow 1^-} \frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] = 0. \quad (25)$$

Proof. (ii) $\Rightarrow$ (i). This implication is obvious.

(i) $\Rightarrow$ (iii). Suppose that $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_{\mu,0}$ is bounded. Let $f(z) = z$ and $f(z) = z^2$, we obtain (23). Since $C_\varphi^g : \mathcal{N}_\alpha^p \rightarrow \mathcal{Z}_\mu$ is bounded, by Corollary 6, we conclude that (21) and (22) hold. Thus for any $\varepsilon > 0$, there exists a $t \in (0, 1)$, such that

$$\frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \varepsilon \quad (26)$$

whenever $t < |\varphi(z)| < 1$. Moreover, using (23), then there exists an $r \in (0, 1)$ such that for $r < |z| < 1$,

$$\mu(|z|)|g'(z)| < \varepsilon(1 - t^2) \exp \left[\frac{-c}{(1 - |t|^2)^{\frac{\alpha+2}{p}}} \right] \quad (27)$$

from which, if $r < |z| < 1$ and $|\varphi(z)| \leq t$, then we have

$$\frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \leq \frac{\mu(|z|)|g'(z)|}{1 - |t|^2} \exp \left[\frac{c}{(1 - |t|^2)^{\frac{\alpha+2}{p}}} \right] < \varepsilon. \quad (28)$$

From (26) and (28), we see that whenever $r < |z| < 1$,

$$\frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] < \varepsilon$$

which implies that (25) holds. Employing (21) and (23), similarly we obtain (24).

(iii) $\Rightarrow$ (ii). Suppose (23), (24) and (25) hold for all $c > 0$. Let $f \in \mathcal{N}_\alpha^p$, by Lemma 1, we have

$$\begin{aligned} \mu(|z|)|(C_\varphi^g f)''(z)| &\leq \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \exp \left[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right] \\ &\quad + \frac{\mu(|z|)|g'(z)|}{1 - |\varphi(z)|^2} \exp \left[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1 - |\varphi(z)|^2)^{\frac{\alpha+2}{p}}} \right]. \end{aligned}$$

Taking the supremum in this inequality over all $f \in \mathcal{N}_\alpha^p$ such that $\|f\|_{\mathcal{N}_\alpha^p} \leq 1$, applying (24) and (25) we obtain $\lim_{|z| \rightarrow 1^-} \sup_{\|f\|_{\mathcal{N}_\alpha^p} \leq 1} \mu(|z|)|(C_\varphi^g f)''(z)| = 0$. The result follows from Lemma 2. $\square$

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