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# Conditions for Positivity of Operators in Non-unital C*-algebras

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**Abstract**

In this paper, we present results on the necessary and sufficient conditions for positivity of operators in non-unital C*- algebras.

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**Keywords:** C*-algebra, positivity, non-unital C*-algebra, Hilbert space

## 1 Introduction

In this paper, we present some important results pertaining to the necessary and sufficient conditions for positive operators in non-unital C*-algebras. Throughout the paper, by $\mathcal{C}^*_{NU}$ we mean non-unital C*-algebras and $\tilde{\mathcal{C}}^*_{NU}$, their unitization.

**Definition 1.1.** A C*- algebra $\mathcal{A}$ is said to be *unital* or *have a unit* $I$ if it has an element, denoted by $I$, satisfying $IA = AI = A \; \forall \; A \in \mathcal{A}$. The element $I$ is called the *multiplicative identity.*

**Definition 1.2.** A C*- algebra $\mathcal{C}^*_{NU}$ is said to be *non-unital* if it does not admit a multiplicative identity $I$.

**Definition 1.3.** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *norm-attainable* if there exists a unit vector $x \in \mathcal{H}$, such that $\|Tx\| = \|T\|$.

# 2 Preliminary

**Lemma 2.1.** *Let $\mathcal{C}^*_{NU}$ be a non-unital C*-algebra, let $\tilde{\mathcal{C}}^*_{NU}$ be the unitization of $\mathcal{C}^*_{NU}$, and let $\varphi : \mathcal{C}^*_{NU} \to \mathbb{C}$ be a positive linear functional. Then $\varphi$ has a unique positive extension $\tilde{\varphi} : \tilde{\mathcal{C}}^*_{NU} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

*Proof.* Assume that $\tilde{\varphi} : \tilde{\mathcal{C}}^*_{NU} \to \mathbb{C}$ is a positive extension of $\varphi$ such that $\|\varphi\| = \|\tilde{\varphi}\|$, then $\tilde{\varphi}(I) = \|\tilde{\varphi}\| = \|\varphi\|$. Now, define $\tilde{\varphi}$ by $\tilde{\varphi}(\lambda I + A) = |\lambda|\|\tilde{\varphi}(I)\| + \|\tilde{\varphi}(A)\| = \lambda\|\varphi\| + \|\varphi(A)\|$. Then, if there is a norm-preserving positive extension of $\varphi$ it must be unique.
To show that $\tilde{\varphi}$ is positive we need to show that $\|\tilde{\varphi}\| = \tilde{\varphi}(I)$. Let $(E_\lambda)_\Lambda$ be a C*-bounded approximate identity for $\mathcal{A}$. Since $\varphi$ is positive, then $\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$. Since $\varphi$ is positive, we have for all $\alpha I + A \in \tilde{\mathcal{C}}^*_{NU}$ that

$$\begin{aligned}
|\tilde{\varphi}(\alpha I + A)| &= |\tilde{\varphi}(\alpha I) + \tilde{\varphi}(A)| \\
&= |\alpha\tilde{\varphi}(I) + \tilde{\varphi}(A)| \\
&= |\alpha\|\varphi\| + \|\varphi(A)\|| \\
&= \lim_{\Lambda} |\alpha\varphi(E_\lambda) + \varphi(AE_\lambda)| \\
&= \lim_{\Lambda} |\varphi(\alpha E_\lambda + AE_\lambda)| \\
&\leq \lim\sup_{\Lambda} \|\varphi\| \, \|(\alpha I + A) E_\lambda\| \\
&\leq \lim\sup_{\Lambda} \|\varphi\| \, \|(\alpha I + A)\| \, \|E_\lambda\| \\
&= \|\varphi\| \, \|(\alpha I + A)\|
\end{aligned}$$

Hence $\|\tilde{\varphi}\| \leq \|\varphi\|$. Since, $\|\varphi\| \geq \|\tilde{\varphi}\|$, the result follows. □

**Theorem 2.2.** *Let $\mathcal{C}^*_{NU} \subseteq \mathcal{B}$ be a non-unital C*-algebra and let $\varphi : \mathcal{C}^*_{NU} \to \mathbb{C}$ be a positive operator. Then there exists a positive linear operator $\psi : \mathcal{B} \to \mathbb{C}$ such that $\psi|_{\mathcal{C}^*_{NU}} = \varphi$ and $\|\psi\| = \|\varphi\|$.*

*Proof.* Let $\tilde{\mathcal{B}}$ be the unitization of $\mathcal{B}$ if $\mathcal{B}$ is non-unital. Consider the *-algebra, $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU} = \{\lambda I_{\tilde{\mathfrak{B}}} + A : A \in \mathcal{C}^*_{NU}, \, \lambda \in \mathbb{C}\} \subseteq \tilde{\mathfrak{B}}$. Let $\tilde{\mathcal{C}}^*_{\mathcal{NU}}$ be the unitization

of $\mathcal{C}^*_{NU}$ and define $\pi : \tilde{\mathcal{C}}^*_{\mathcal{NU}} \to \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU}$ by $\pi(\lambda I_{\tilde{\mathcal{C}}^*_{\mathcal{NU}}} + A) = \lambda I_{\tilde{\mathfrak{B}}} + A$. Hence, $\pi$ is a *-homomorphism. As the domain of $\pi$ is a C*-algebra and the range of $\pi$ is embedded inside the C*-algebra $\tilde{\mathcal{B}}$, the range of $\pi$ is a C*-algebra as $\pi(\tilde{\mathcal{C}}^*_{\mathcal{NU}}) = \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU}, \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU}$ is a C*-subalgebra of $\tilde{\mathfrak{B}}$ with the same unit. Moreover, if $\pi(\lambda I_{\tilde{\mathcal{C}}^*_{\mathcal{NU}}} + A) = 0$ then $\lambda I_{\tilde{\mathfrak{B}}} = -A \in \mathcal{C}^*_{NU}$. As $\mathcal{C}^*_{NU}$ is non-unital, this implies that $\lambda = 0$ and hence $A = 0$. Therefore, $\pi$ must be injective and thus $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU}$ is *-isomorphic to $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU}$.
By Lemma 2.1, $\varphi$ extends to a positive linear functional $\tilde{\varphi} : \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$. Since $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathcal{C}^*_{NU} \subseteq \tilde{\mathfrak{B}}$ are C*-algebras with the same unit, $\tilde{\varphi}$ extends to a positive linear functional $\tilde{\psi} : \tilde{\mathfrak{B}} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\tilde{\psi}\|$. Let $\psi : \mathcal{B} \to \mathbb{C}$ be defined by $\psi = \tilde{\psi}|_{\mathcal{B}}$. Since the restriction of a positive linear operator is clearly positive, $\psi$ is a positive linear operator. Moreover, $\psi$ extends $\varphi$ and $\|\psi\| \leq \|\tilde{\psi}\| = \|\varphi\| \leq \|\psi\|$. □

# 3 Main Results

**Lemma 3.1.** *Let $T \in \mathcal{C}^*_{NU}$, then the operator $T$ is positive if it is normal and self adjoint. Moreover, it is completely positive if $T$ is norm-attainable.*

*Proof.* Clearly, $\|T\| \geq 0, \forall T \in \mathcal{C}^*_{NU}$. Let $T^* : H \to H$ be the adjoint of $T$. Then clearly since $T$ is a bounded linear operator, it commutes with its adjoint i.e. $T^*T = TT^*$ hence normal. Also the norm of $T$ is equal to the norm of $T^*$ i.e. $\|T\| = \|T^*\|$.
Now, let $T$ be completely positive. Define $T_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B})$ by, $T_n(A_{ij}) = [T_n(A_{ij})]$, then $\lim_{n\to\infty} \|T_n(A_{ij})\| = \|T(A_{ij})\| = \|T\|$ since $\|A_{ij}\| = 1$. Hence $T$ is norm-attainable. □

**Corollary 3.2.** *Let $T \in \mathcal{C}^*_{NU}$, then the following properties are equivalent.*

*(i) $T$ is normal.*

*(ii) $T$ is norm-attainable.*

*(iii) $T$ is positive.*

*Proof.* $(1 \Rightarrow 2)$ Let $T \in \mathcal{C}^*_{NU}$ be a normal operator, then there exists a unit vector $x \in \mathcal{H}$ such that $\|Tx\| = \|T\|$. Hence $T$ is norm-attainable.
$(2 \Rightarrow 3)$ If $T$ is norm-attainable, then by Lemma 3.1 it is completely positive hence positive.
$(3 \Rightarrow 1)$ Let $T$ be positive, then $\|T\| \geq 0, \forall, T \in \mathcal{C}^*_{NU}$. Let $T^* : H \to H$ be the adjoint of $T$. Then as $T$ is a bounded linear operator, it commutes with its adjoint i.e. $T^*T = TT^*$ hence normal. □

Next we characterize convergence of positive elements in a non-unital C*-algebra.

**Theorem 3.3.** *Let $\mathcal{C}^*_{NU}$ be a non-unital C*-algebra. Suppose that $(\varphi_m)_{m\geq 1} \in \mathcal{C}^*_{NU}$ is a sequence such that $\lim_{n\to\infty} \varphi_m = \varphi \in \mathcal{C}^*_{NU}$ and $\varphi_m \geq 0$ for all $m \in \mathbb{N}$, then $\varphi$ is positive, self-adjoint and normal.*

*Proof.* Let $\tilde{\mathcal{C}}^*_{NU}$ be the unitization of $\mathcal{C}^*_{NU}$. By the continuity of the adjoint,

$$\varphi^* = \lim_{m\to\infty} \varphi^*_m = \lim_{m\to\infty} \varphi_m = \varphi$$

showing that $\varphi_m$ is self-adjoint.
Let $C = \sup_{m\geq 1} \|\varphi_m\| < \infty$, then $\|\varphi\| \leq C$. Since $0 \leq \varphi_m \leq CI$ for all $m$, $0 \leq 2\varphi_m \leq 2CI$ for all $m$ and thus $-CI \leq 2\varphi_m - CI \leq 2CL$ for all $m$. Thus by the Continuous Functional Calculus, $\|2\varphi_m - CI\| \leq C$ for all $m$. Since $\lim_{n\to\infty} \varphi_m = \varphi$, $\lim_{n\to\infty} 2\varphi_m - CI = 2\varphi - CI$. So, $\|2\varphi_m - CI\| \leq C$. Hence, $-CI \leq 2\varphi_m - CI \leq 2CL$, thus $0 \leq \varphi \leq CI$. Therefore $\varphi$ is positive as required. □

# 4 Conclusion

In this paper, we have established the necessary and sufficient conditions for positivity of operators in non-unital C*-algebras. The question which arises is; Are positive operators in non-unital C*-algebras completely positive?.

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