International Journal of Mathematical Analysis
Vol. 9, 2015, no. 3, 101 - 110
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ijma.2015.411337

New Fuzzy Measure Based on Gradual Numbers

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Abstract

In this paper, we deal with special generalization of measures by considering their values in the set of gradual numbers. Firstly, the concept of gradual number-valued measures is introduced and some of its properties are investigated. And then, Lebesgue type decomposition theorem for this kind of measure is obtained.

Mathematics Subject Classification: 28E10, 03E72

Keywords: Gradual number; measure; decomposition theorem

1 Introduction

Motivated by the applications in several areas of applied science, such as mathematical economics, fuzzy optimal, process control and decision theory, much effort has been devoted to the generalization of different measure concepts and classical results to the case when outcomes of a random experiment are represented by fuzzy sets, such as the concept of fuzzy measures [2, 18–21], the concept of fuzzy random variables [8, 11–13], expected values and variances of fuzzy random variables [4]. In above literatures, the most often proposed fuzzy sets are fuzzy numbers [5] or fuzzy real numbers [9]. But it is well known that fuzzy numbers generalize intervals, not numbers and model incomplete knowledge, not the fuzziness per se. Furthermore, fuzzy arithmetics inherit algebraic properties of interval arithmetics, not of numbers. In particular, addition does not define a group on such fuzzy numbers. There are some authors who tried to equip fuzzy numbers with a group structure. This kind of attempt suggests a confusion between uncertainty and gradualness. Hence the name "fuzzy number" used by many authors is debatable. On the other hand, a different view of a fuzzy real number, starting with Hutton [9], which is usual defined as a non-increasing upper-semicontinuous function $\tilde{z} : \mathbb{R} \to [0, 1]$ such that $\sup \tilde{z}(x) = 1$ and $\inf \tilde{z}(x) = 0$, the arithmetic operations are also defined by the extension principle. It is much more complicated than ordinary arithmetics. In particular what concerns inverse operations like subtraction and division.

In order to model the essence of fuzziness without uncertainty, the concept of gradual numbers has been proposed by Fortin *et al.* [6]. Gradual numbers are unique generalization of the real numbers, which are equipped with the same algebraic structures as real numbers (addition is a group, etc.). The authors claimed that the concept of gradual numbers is a missing primitive concept in fuzzy set theory. In the brief time since their introduction, gradual numbers have been employed as tools for computations on fuzzy intervals, with applications to combinatorial fuzzy optimization [7,10] and others [1,3,6,14–16,22]. Hence, it is an interesting question whether a measure can be introduced with value in the set of gradual numbers. This is the motivation of current work. In this paper, the concept of gradual number-valued measures is introduced. Some of its properties and structural characterizations are investigated.

The organization of the paper is as follows. In Section 2, we state some basic concepts and preliminary results about gradual numbers. In Section 3, the concept of gradual number-valued measures is introduced. And then, some of its properties are investigated. Finally, Lebesgue type decomposition theorem for this kind of measure is obtained.

2 Preliminaries

We state some basic concepts about gradual numbers. All concepts and signs not defined in this paper may be found in [6, 17, 22].

Definition 2.1 *[6] A gradual real number (or gradual number for short) $\tilde{r}$ is defined by an assignment function*

$$A_{\tilde{r}} : (0, 1] \to \mathbb{R}.$$

Naturally a nonnegative gradual number is defined by its assignment function from $(0,1]$ *to* $[0,+\infty)$.

In the following, $\tilde{r}(\alpha)$ may be substituted for $A_{\tilde{r}}(\alpha)$. The set of all gradual numbers (resp. nonnegative gradual numbers) is denoted by $\mathbb{R}(I)$ (resp. $\mathbb{R}^*(I)$).

A crisp element $b \in \mathbb{R}$ has its own assignment function $\tilde{b} : (0,1] \to \mathbb{R}$ defined by

$$\tilde{b}(\alpha) = b, \forall \alpha \in (0,1].$$

We call such elements in $\mathbb{R}(I)$ constant gradual numbers. In particular, $\tilde{0}$ (resp. $\tilde{1}$) denotes constant gradual number defined by $\tilde{0}(\alpha) = 0$ (resp. $\tilde{1}(\alpha) = 1$) for all $\alpha \in (0,1]$. Obviously, there is a bijection between the set of all constant gradual numbers and $\mathbb{R}$. Hence $\mathbb{R}$ can be regarded as a subset of $\mathbb{R}(I)$ if we do not distinguish $\tilde{b}$ and b.

Definition 2.2 *[6] Let* $\tilde{r}, \tilde{s} \in \mathbb{R}(I)$ *and* $\gamma \in \mathbb{R}$. *The operations are defined as follows:*

(1) $(\tilde{r} + \tilde{s})(\alpha) = \tilde{r}(\alpha) + \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(2) $(\tilde{r} - \tilde{s})(\alpha) = \tilde{r}(\alpha) - \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(3) $(\tilde{r} \cdot \tilde{s})(\alpha) = \tilde{r}(\alpha) \cdot \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(4) $\left(\frac{\tilde{r}}{\tilde{s}}\right)(\alpha) = \frac{\tilde{r}(\alpha)}{\tilde{s}(\alpha)}$, *if* $\tilde{s}(\alpha) \neq 0, \forall \alpha \in (0,1]$.

Definition 2.3 *[17] Let* $\tilde{r}, \tilde{s} \in \mathbb{R}(I)$. *The relations between* $\tilde{r}$ *and* $\tilde{s}$ *are defined as follows:*

(1) $\tilde{r} = \tilde{s}$ *if and only if* $\tilde{r}(\alpha) = \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(2) $\tilde{r} \succeq \tilde{s}$ *if and only if* $\tilde{r}(\alpha) \geq \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(3) $\tilde{r} \preceq \tilde{s}$ *if and only if* $\tilde{r}(\alpha) \leq \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(4) $\tilde{r} \succ \tilde{s}$ *if and only if* $\tilde{r}(\alpha) > \tilde{s}(\alpha), \forall \alpha \in (0,1]$;
(5) $\tilde{r} \prec \tilde{s}$ *if and only if* $\tilde{r}(\alpha) < \tilde{s}(\alpha), \forall \alpha \in (0,1]$.

Definition 2.4 *[22] Let* $\{\tilde{r}_n\} \subseteq \mathbb{R}(I)$ *and* $\tilde{r} \in \mathbb{R}(I)$.

(1) $\{\tilde{r}_n\}$ *is said to converge to* $\tilde{r}$ *if for each* $\alpha \in (0,1]$,

$$\lim_{n\to\infty} \tilde{r}_n(\alpha) = \tilde{r}(\alpha)$$

and it is denoted as $\lim_{n\to\infty} \tilde{r}_n = \tilde{r}$ *or* $\tilde{r}_n \to \tilde{r}$, *if there is no such an* $\tilde{r}$, *the sequence* $\{\tilde{r}_n\}$ *is said to be divergent;*

(2) *if* $\lim_{n\to\infty} \sum_{i=1}^{n} \tilde{r}_i$ *exists, then the infinite sum of sequence* $\{\tilde{r}_n\}$ *is defined by*

$$\sum_{i=1}^{\infty} \tilde{r}_i = \lim_{n\to\infty} \sum_{i=1}^{n} \tilde{r}_i,$$

if $\lim_{n\to\infty} \sum_{i=1}^{n} \tilde{r}_i$ *does not exist, then the infinite sum of sequence* $\{\tilde{r}_n\}$ *is said to be divergent.*

3 Main Results

In order to generalize the notion of measure value to gradual numbers, we can naturally define as follows:

Definition 3.1 *Let $(X, \mathscr{A})$ be a measurable space. A mapping $\tilde{m} : \mathscr{A} \to \mathbb{R}^*(I)$ is called a gradual number-valued measure if it satisfies the following two conditions:*

(1) $\tilde{m}(\emptyset) = \tilde{0}$;

(2) *if $A_1, A_2, \ldots$ are in $\mathscr{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, then*

$$\tilde{m}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \tilde{m}(A_i).$$

The second condition is called countable additivity of the gradual number-valued measure $\tilde{m}$. We say that $(X, \mathscr{A}, \tilde{m})$ is a gradual number-valued measure space.

Evidently, a classical measure must be a special gradual number-valued measure if $\mathbb{R}$ is regarded as a subset of $\mathbb{R}(I)$.

In the following, we give two examples in order to show the existence of such measure spaces.

Example 3.2 *Let $\mu_1, \mu_2, \ldots, \mu_n$ be classical measures defined on the same measurable space $(X, \mathscr{A})$ and $\tilde{m} : \mathscr{A} \to \mathbb{R}^*(I)$ a mapping defined by*

$$\tilde{m}(A)(\alpha) = \alpha \sum_{i=1}^{n} \mu_i(A), \forall A \in \mathscr{A}, \forall \alpha \in (0, 1].$$

Then it is easy to prove that $(X, \mathscr{A}, \tilde{m})$ is a gradual number-valued measure space.

Example 3.3 *Let X be a nonempty finite set. For each $A \in 2^X$, we define $\tilde{r}_A : (0, 1] \to [0, +\infty)$ by*

$$\tilde{r}_A(\alpha) = \begin{cases} |A| & if\ \alpha = 1, \\ 0 & otherwise, \end{cases}$$

and $\tilde{m} : 2^X \to \mathbb{R}^*(I)$ by

$$\tilde{m}(A) = \tilde{r}_A, \forall A \in \mathscr{A}.$$

Then $(X, 2^X, \tilde{m})$ is a gradual number-valued measure space.

Theorem 3.4 *Let $(X, \mathscr{A}, \tilde{m})$ be a gradual number-valued measure space.*

(1) *If $A_1, A_2, \ldots, A_n$ are in $\mathscr{A}$, with $A_i \cap A_j = \emptyset$ for $i \neq j$, then* $\tilde{m}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \tilde{m}(A_i)$;

(2) *if A, B are in $\mathscr{A}$ and $A \subseteq B$, then* $\tilde{m}(B \setminus A) = \tilde{m}(B) - \tilde{m}(A)$ *and* $\tilde{m}(B) \succeq \tilde{m}(A)$;

(3) *if A, B are in $\mathscr{A}$, then* $\tilde{m}(A \cup B) + \tilde{m}(A \cap B) = \tilde{m}(A) + \tilde{m}(B)$;

(4) *if $A_1, A_2, \ldots$ are in $\mathscr{A}$, then* $\tilde{m}(\bigcup_{i=1}^{\infty} A_i) \preceq \sum_{i=1}^{\infty} \tilde{m}(A_i)$.

Proof. (1) Since $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} A_i \bigcup \emptyset \bigcup \emptyset \cdots$, it follows from Definition 3.1 that

$$\tilde{m}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \tilde{m}(A_i).$$

(2) Since $A \subseteq B$, we have

$$B = A \cup (B \setminus A), A \cap (B \setminus A) = \emptyset.$$

By conclusion (1), we obtain that $\tilde{m}(B) = \tilde{m}(A) + \tilde{m}(B \setminus A)$. This implies that $\tilde{m}(B \setminus A) = \tilde{m}(B) - \tilde{m}(A)$. In addition, $\tilde{m}(B \setminus A) \succeq \tilde{0}$ implies that $\tilde{m}(B) - \tilde{m}(A) \succeq \tilde{0}$, and hence $\tilde{m}(B) \succeq \tilde{m}(A)$.

(3) On the one hand, since $A \cup B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \emptyset$, we obtain that

$$\tilde{m}(A \cup B) = \tilde{m}(A) + \tilde{m}(B \setminus A).$$

On the other hand, since $A \cap B = B \setminus (B \setminus A)$ and $(B \setminus A) \subseteq B$, by conclusion (2), we have

$$\tilde{m}(A \cap B) = \tilde{m}(B) - \tilde{m}(B \setminus A).$$

Combining above two equalities, we get

$$\tilde{m}(A \cup B) + \tilde{m}(A \cap B) = \tilde{m}(A) + \tilde{m}(B).$$

(4) Note that the family of sets

$$B_1 = A_1,\ B_2 = A_2 \setminus A_1,\ B_3 = A_3 \setminus (A_1 \cup A_2), \ldots,\ B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$$

is disjoint and $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$. Moreover, since $B_i \subseteq A_i$, we have that

$$\tilde{m}(B_i) \preceq \tilde{m}(A_i).$$

Hence

$$\tilde{m}\left(\bigcup_{i=1}^{n} A_i\right) = \tilde{m}\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \tilde{m}(B_i) \preceq \sum_{i=1}^{n} \tilde{m}(A_i).$$

Now we can repeat the argument for the infinite family using countable additivity of gradual number-valued measure. This completes the proof.

Theorem 3.5 *Let $(X, \mathscr{A})$ be a measurable space and $\tilde{m} : \mathscr{A} \to \mathbb{R}^*(I)$ a mapping. Define $\tilde{m}_\alpha : \mathscr{A} \to [0, +\infty)$ by*

$$\tilde{m}_\alpha(A) = \tilde{m}(A)(\alpha), \forall A \in \mathscr{A}.$$

Then $\tilde{m}$ is a gradual number-valued measure if and only if $\{\tilde{m}_\alpha : \alpha \in (0, 1]\}$ is a family of classical measures on $(X, \mathscr{A})$.

Proof. Necessity. Suppose that $\tilde{m}$ is a gradual number-valued measure. According to Definition 3.1, for each $\alpha \in (0,1]$, the following two conclusions hold:

(1) $\tilde{m}_\alpha(\emptyset) = \tilde{m}(\emptyset)(\alpha) = \tilde{0}(\alpha) = 0$;

(2) if $A_1, A_2, \ldots$ be in $\mathscr{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\tilde{m}_\alpha\left(\bigcup_{i=1}^{\infty} A_i\right) = \tilde{m}\left(\bigcup_{i=1}^{\infty} A_i\right)(\alpha) = \left(\sum_{i=1}^{\infty} \tilde{m}(A_i)\right)(\alpha) = \sum_{i=1}^{\infty} \tilde{m}_\alpha(A_i).$$

Therefore, for each $\alpha \in (0,1]$, $\tilde{m}_\alpha$ is a classical measure.

Sufficiency. If $\{\tilde{m}_\alpha : \alpha \in (0,1]\}$ is a family of classical measures on $(X, \mathscr{A})$, then

(1) for each $\alpha \in (0,1]$, we have

$$\tilde{m}(\emptyset)(\alpha) = \tilde{m}_\alpha(\emptyset) = 0.$$

This implies that $\tilde{m}(\emptyset) = \tilde{0}$.

(2) For each $\alpha \in (0,1]$,

$$\tilde{m}\left(\bigcup_{i=1}^{\infty} A_i\right)(\alpha) = \tilde{m}_\alpha\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \tilde{m}_\alpha(A_i) = \left(\sum_{i=1}^{\infty} \tilde{m}(A_i)\right)(\alpha).$$

This implies that

$$\tilde{m}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \tilde{m}(A_i).$$

Hence $\tilde{m}$ is a gradual number-valued measure. This completes the proof.

Theorem 3.6 *Let $(X, \mathscr{A}, \tilde{m})$ be a gradual number-valued measure space. Then*

(1) *$\tilde{m}$ is continuous from below, i.e., $\tilde{m}(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \tilde{m}(A_n)$ if $\{A_n\}$ is a monotonically increasing sequence of sets of $\mathscr{A}$;*

(2) *$\tilde{m}$ is continuous from above, i.e., $\tilde{m}(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \tilde{m}(A_n)$ if $\{A_n\}$ is a monotonically decreasing sequence of sets of $\mathscr{A}$.*

Proof. This follows immediately from Theorem 3.5.

Corollary 3.7 *Let $(X, \mathscr{A}, \tilde{m})$ be a gradual number-valued measure space. Then $\tilde{m}$ is continuous in $\emptyset$, i.e., if $\{A_n\}$ is a monotonically decreasing sequence of sets of $\mathscr{A}$ such that $A_n \downarrow \emptyset$, then $\lim\limits_{n\to\infty} \tilde{m}(A_n) = \tilde{0}$.*

Theorem 3.8 *Let $(X, \mathscr{A})$ be a measurable space. A mapping $\tilde{m} : \mathscr{A} \to \mathbb{R}^*(I)$ is the gradual number-valued measure if and only if $\tilde{m}$ satisfies the following conditions:*

(1) *If A and B are in $\mathscr{A}$ with $A \cap B = \emptyset$, then $\tilde{m}(A \cup B) = \tilde{m}(A) + \tilde{m}(B)$;*

(2) *$\tilde{m}$ is continuous from below.*

Proof. Necessity is obvious. In the following we prove the sufficiency. Suppose that A and B are in $\mathscr{A}$ with $A \cap B = \emptyset$, then $\tilde{m}(A \cup B) = \tilde{m}(A) + \tilde{m}(B)$. It follows that for each $\alpha \in (0, 1]$,

$$\tilde{m}_\alpha(A \cup B) = \tilde{m}_\alpha(A) + \tilde{m}_\alpha(B).$$

Also, since $\tilde{m}$ is continuous from below, which implies that for each $\alpha \in (0, 1]$,

$$\tilde{m}_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \tilde{m}_\alpha(A_n)$$

if $A_1, A_2, \ldots$ are in $\mathscr{A}$ a monotonically decreasing sequence of sets. Then we can conclude that for each $\alpha \in (0, 1]$, $\tilde{m}_\alpha$ is a classical measure. By Theorem 3.5, we know that $\tilde{m}$ is a gradual number-valued measure. This completes the proof.

In the last of this section, we give the Lebesgue type decomposition theorem for gradual number-valued measure.

Definition 3.9 *Let $(X, \mathscr{A}, m)$ be a classical measure space and $\tilde{m}$ a gradual number-valued measure on $(X, \mathscr{A})$. We call that $\tilde{m}$ is continuous absolutely about m if for arbitrary $A \in \mathscr{A}$, $m(A) = 0$, then $\tilde{m}(A) = \tilde{0}$, denoted as $\tilde{m} \ll m$. We call that $\tilde{m}$ is singular about m if there exist $A, B \in \mathscr{A}$ such that for arbitrary $A \in \mathscr{A}$, $\tilde{m}(A \cap N^c) = \tilde{0}$, denoted as $\tilde{m} \perp m$.*

Theorem 3.10 *Let $(X, \mathscr{A}, m)$ be a classical measure space and $\tilde{m}$ a gradual number-valued measure on $(X, \mathscr{A})$. Then*

(1) $\tilde{m}$ is continuous absolutely about m if and only if for all $\alpha \in (0, 1]$, $\tilde{m}_\alpha$ is continuous absolutely about m;

(2) $\tilde{m}$ is singular about m if and only if for all $\alpha \in (0, 1]$, $\tilde{m}_\alpha$ is singular about m.

Proof. (1) Firstly, suppose that $\tilde{m} \ll m$. For arbitrary $A \in \mathscr{A}$, if $m(A) = 0$, then $\tilde{m}(A) = \tilde{0}$. It follows that for each $\alpha \in (0, 1]$, $\tilde{m}_\alpha(A) = 0$. This implies that $\tilde{m}_\alpha \ll m$.

Conversely, suppose that for all $\alpha \in (0, 1]$, we have $\tilde{m}_\alpha \ll m$. For arbitrary $A \in \mathscr{A}$, if $m(A) = 0$, then for all $\alpha \in (0, 1]$, $\tilde{m}_\alpha(A) = \tilde{m}(A)(\alpha) = 0$. This implies that $\tilde{m}(A) = \tilde{0}$. Hence, $\tilde{m} \ll m$.

(2) Suppose that $\tilde{m}$ is singular about m. Then there exists $N \in \mathscr{A}$ such that for arbitrary $A \in \mathscr{A}$, we have $\tilde{m}(A \cap N^c) = \tilde{0}$. This follows that for each $\alpha \in (0, 1]$, $\tilde{m}_\alpha(A \cap N^c) = 0$, i.e., for each $\alpha \in (0, 1]$, $\tilde{m}_\alpha \perp m$.

Conversely, suppose that for all $\alpha \in (0, 1]$, $\tilde{m}_\alpha \perp m$. Then for each $\alpha \in (0, 1]$, there exists $N_\alpha \in \mathscr{A}$ such that for arbitrary $A \in \mathscr{A}$, $\tilde{m}_\alpha(A \cap N_\alpha^c) = 0$. Let $N = \cup_{\alpha\in(0,1]} N_\alpha$. Then for arbitrary $A \in \mathscr{A}$,

$$A \cap N^c = A \cap (\cap_{\alpha\in(0,1]} N_\alpha^c) \subseteq A \cap N_\alpha^c.$$

It follows that

$$\tilde{m}_\alpha(A \cap N^c) \leq \tilde{m}_\alpha(A \cap N^c_\alpha) = 0$$

for each $\alpha \in (0,1]$. This implies that $\tilde{m}_\alpha(A \cap N^c) = 0$ for each $\alpha \in (0,1]$. Hence $\tilde{m}(A \cap N^c) = \tilde{0}$. This completes the proof.

Theorem 3.11 *(Generalization of Lebesgue decomposition theorem for gradual number-valued measure) Let $(X, \mathscr{A}, m)$ be a classical measure space and $\tilde{m}$ is gradual number-valued measure on $(X, \mathscr{A})$. Then there exist two gradual number-valued measures $\tilde{m}_a$ and $\tilde{m}_s$ such that*

$$\tilde{m} = \tilde{m}_a + \tilde{m}_s,\ \tilde{m}_a \ll m,\ \tilde{m}_s \perp m.$$

Proof. According to Theorem 3.5, if $\tilde{m}$ is a gradual number-valued measure, then for each $\alpha \in (0,1]$, $\tilde{m}_\alpha$ is a classical measure. By classical Lebesgue decomposition theorem, there exist two classical measures $(\tilde{m}_\alpha)_a$ and $(\tilde{m}_\alpha)_s$ such that

$$\tilde{m}_a = (\tilde{m}_\alpha)_a + (\tilde{m}_\alpha)_s,\ (\tilde{m}_\alpha)_a \ll m,\ (\tilde{m}_\alpha)_s \perp m.$$

We build up two gradual number-valued set functions $\tilde{m}_a, \tilde{m}_s : \mathscr{A} \to \mathbb{R}(I)$ such that

$$\tilde{m}_a(A)(\alpha) = (\tilde{m}_\alpha)_a(A),\ \tilde{m}_s(A)(\alpha) = (\tilde{m}_\alpha)_s(A), \forall \alpha \in (0,1], \forall A \in \mathscr{A}.$$

Then we can prove that $\tilde{m}_a$ and $\tilde{m}_s$ are gradual number-valued measures such that $\tilde{m} = \tilde{m}_a + \tilde{m}_s$. Furthermore,

$$(\tilde{m}_a)_\alpha = (\tilde{m}_\alpha)_a,\ (\tilde{m}_s)_\alpha = (\tilde{m}_\alpha)_s.$$

According to Theorem 3.10, we have $\tilde{m}_a \ll m,\ \tilde{m}_s \perp m$. This completes the proof.

Acknowledgements. The project is supported by the National Natural Science Foundation of China (41201327), Natural Science Foundation of Hebei Province (A2013201119), Education Department of Hebei Province (QN20131055) and Science and Technology Bureau of Baoding City (14ZF058). Their financial support is gratefully acknowledged.

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Received: November 12, 2014; Published: January 9, 2015