Stability of an $n$-Dimensional Functional Equation Related to Quadratic-Additive Mappings in Fuzzy Normed Spaces

Sun Sook Jin
Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Republic of Korea

Yang-Hi Lee
Department of Mathematics Education
Gongju National University of Education
Gongju 314-711, Republic of Korea

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Abstract

In this paper, we investigate a fuzzy version of stability for the functional equation

$$\sum_{i=1}^{n} f \left( x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right) - \sum_{i=1}^{n} f(x_i) + nf \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) = 0$$

whose solutions are quadratic-additive mappings, in the sense of A. K. Mirmostafaei and M. S. Moslehian.

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1 Introduction and preliminaries

In 1984, A. K. Katsaras [12] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, some mathematicians have introduced several types of fuzzy norm in different points of view. In 2003, T. Bag and S.K. Samanta [2] gave a new definition of a fuzzy norm to exhibit a reasonable fuzzy version of stability for functional equations by following:

Definition 1.1 ([2]) Let $X$ be a real linear space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$\text{(N1)}$ $N(x, c) = 0$ for $c \leq 0$;
$\text{(N2)}$ $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;
$\text{(N3)}$ $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;
$\text{(N4)}$ $N(x + y, s + t) \geq \min \{N(x, s), N(y, t)\}$;
$\text{(N5)}$ $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$.

A classical stability problem of the functional equation was formulated by S. M. Ulam [24] in 1940. In the next year, D. H. Hyers [5] gave a partial solution of Ulam’s problem. Subsequently, his result was generalized by T. Aoki [1] and Th. M. Rassias [22], and then the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [3, 4, 11, 14, 15, 16, 23].


$$f(x + y) - f(x) - f(y) = 0$$ (1)

and the quadratic functional equation:

$$f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0$$ (2)

where every solution of (1) is called an additive mapping and every solution of (2) is called a quadratic mapping. On the other hand if a mapping is represented by sum of an additive mapping and a quadratic mapping, we call the mapping a quadratic-additive mapping.

In this paper, we get a stability result of the following $n$-dimensional functional equation

$$\sum_{i=1}^{n} f \left( x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right) - \sum_{i=1}^{n} f(x_i) + nf \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right) = 0$$ (3)

in the fuzzy normed linear spaces.
In 2008, C.-G. Park et al [21] showed that every solution of (3) is a quadratic-additive mapping. In 2011, Z. Wang et al [25] obtained a stability of the functional equation (3) in fuzzy spaces. In their study [25] of stability problem of (3), they attempted to get stability theorems to take the desired approximate solution by handling the odd and even part of \( f \), respectively. In this paper, however, we can take the desired approximate solution at once. Therefore, this idea is a refinement with respect to the simplicity of the proof. Using the similar method, the authors and S.-M. Jung showed the stability of various kinds of functional equations related to quadratic-additive mappings in fuzzy spaces, see [6, 7, 8, 9, 18].

2 Stability of the functional equation (3)

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the \( n \)-dimensional functional equation (3) in the fuzzy normed linear space.

Let \((X, N)\) be a fuzzy normed linear space. Let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is said to be convergent if there exists \(x \in X\) such that \(\lim_{n \to \infty} N(x_n - x, t) = 1\) for all \(t > 0\). In this case, \(x\) is called the limit of the sequence \(\{x_n\}\) and we denote it by \(N \lim_{n \to \infty} x_n = x\). A sequence \(\{x_n\}\) in \(X\) is called Cauchy if for each \(\varepsilon > 0\) and each \(t > 0\) there exists \(n_0\) such that for all \(n \geq n_0\) and all \(p > 0\) we have \(N(x_n + p - x_n, t) > 1 - \varepsilon\). It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Throughout this section, let \((X, N)\) be a fuzzy normed linear space and \((Y, N')\) a fuzzy Banach space. And let \(n\) be a fixed natural number greater than 2. For a given mapping \(f : X \to Y\), we use the abbreviations

\[
Df(x_1, x_2, \cdots, x_n) := \sum_{i=1}^{n} f \left( x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \right) - \sum_{i=1}^{n} f(x_i) + nf \left( \frac{1}{n} \sum_{j=1}^{n} x_j \right)
\]

for all \(x_1, x_2, \cdots, x_n \in X\) and

\[
\Delta(x) := (2x, 2x, -x, x, \cdots, x)
\]

for all \(x \in X\).

**Lemma 2.1** (Lemma 2.1 in [21]) Let \(V\) and \(W\) be real vector spaces. If a mapping \(f : V \to W\) satisfies

\[
Df(x_1, \cdots, x_n) = 0
\]
for all \( x_1, \cdots, x_n \in V \), then the mapping \( f : V \to W \) satisfies
\[
2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y)
\] (4)
for all \( x, y \in V \).

It is easy to show that every solution of the functional equation (4) is a quadratic-additive mapping. Hence, by Lemma 2.1, every solution of the functional equation \( Df(x_1, \cdots, x_n) = 0 \) is a quadratic-additive mapping.

For given \( q > 0 \), the mapping \( f \) is called a fuzzy \( q \)-almost quadratic-additive mapping if
\[
N'(Df(x_1, \cdots, x_n), t_1 + \cdots + t_n) \geq \min\{N(x_1, t_1^q), \cdots, N(x_n, t_n^q)\}
\] (5)
for all \( x_1, x_2, \cdots, x_n \in X \) and \( t_1, t_2, \cdots, t_n \in (0, \infty) \). The following result gives a fuzzy version of the stability of the functional equation (3).

**Theorem 2.2** Let \( q \) be a positive real number with \( q \neq \frac{1}{2}, 1 \). And let \( f \) be a fuzzy \( q \)-almost quadratic-additive mapping from a fuzzy normed space \( (X, N) \) into a fuzzy Banach space \( (Y, N') \). Then there is a unique quadratic-additive mapping \( F : X \to Y \) such that
\[
N'(F(x)-f(x), t) \geq \begin{cases} 
\sup_{t' < t} N\left(x, \frac{t'^q}{2^{(2p-4)t'q}}\right) & \text{if } q > \frac{1}{2}, q \neq 1 \\
\sup_{t' < t} N\left(x, \frac{(2^p-4)t'q}{2^{nq}}\right) & \text{if } 0 < q < \frac{1}{2}
\end{cases}
\] (6)
for each \( x \in X \) and \( t > 0 \), where \( p = 1/q \).

**Proof.** It follows from (N2), (N3), (N4) and (5) that
\[
N'(f(0), t) = N'(Df(0, \cdots, 0), nt) \geq N(0, t^q) = 1
\]
for all \( t > 0 \). So, by (N2), we know that \( f(0) = 0 \). We will prove the theorem in three cases, \( q > 1, \frac{1}{2} < q < 1 \), and \( 0 < q < \frac{1}{2} \).

**Case 1.** Let \( q > 1 \) and let \( J_mf : X \to Y \) be a mapping defined by
\[
J_mf(x) := 2^{-2m-1}(f(2^m x) + f(-2^m x)) + 2^{-m-1}(f(2^m x) - f(-2^m x))
\]
for all \( x \in X \) and \( m \in \mathbb{N} \). Then \( J_0f(x) = f(x) \) and
\[
J_jf(x) - J_{j+1}f(x) = \frac{Df(\Delta(2^j x))}{2 \cdot 4^{j+1}} + \frac{Df(\Delta(-2^j x))}{2 \cdot 4^{j+1}} + \frac{Df(\Delta(2^j x))}{3 \cdot 2^{j+2}} - \frac{Df(\Delta(-2^j x))}{3 \cdot 2^{j+2}}
\] (7)
for all \( x \in X \) and \( j \geq 0 \). Together with (N3), (N4) and (5), this equation implies that if \( m' + m > m \geq 0 \) then

\[
N'(J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \left( \frac{1}{4} \left( \frac{2^p}{4} \right)^j + \frac{1}{6} \left( \frac{2^p}{2} \right)^j \right) n \tilde{t}^p) \\
\geq N' \left( \sum_{j=m}^{m'+m-1} (J_j f(x) - J_{j+1} f(x)), \sum_{j=m}^{m'+m-1} \left( \frac{1}{4} \left( \frac{2^p}{4} \right)^j + \frac{1}{6} \left( \frac{2^p}{2} \right)^j \right) n \tilde{t}^p \right) \\
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N' \left( J_j f(x) - J_{j+1} f(x), \left( \frac{1}{4} \left( \frac{2^p}{4} \right)^j + \frac{1}{6} \left( \frac{2^p}{2} \right)^j \right) n \tilde{t}^p \right) \right\} \\
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ \min \left\{ N' \left( \frac{D f(\Delta(2^j x))}{2 \cdot 4^{j+1}}, \frac{2^j n \tilde{t}^p}{2 \cdot 4^{j+1}} \right), N' \left( \frac{D f(\Delta(2^j x))}{3 \cdot 2^{j+2}}, \frac{2^j n \tilde{t}^p}{3 \cdot 2^{j+2}} \right) \right\} \right\} \\
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N(2^j x, 2^j t), N(2^{j+1} x, 2^j t) \right\} \\
= N \left( x, \frac{\tilde{t}}{2} \right) \\
= N(2^j x, 2^j t), N(2^{j+1} x, 2^j t) \\
= N \left( x, \frac{\tilde{t}}{2} \right)
\]

for all \( x \in X \) and \( t > 0 \). Let \( \varepsilon > 0 \) be given. Since \( \lim_{t \to \infty} N(x, t) = 1 \), there is \( t_0 > 0 \) such that

\[
N(x, t_0) \geq 1 - \varepsilon.
\]

Observe that for some \( \frac{\tilde{t}}{2} > t_0 \), the series \( \sum_{j=0}^{\infty} \left( \frac{1}{4} \left( \frac{2^p}{4} \right)^j + \frac{1}{6} \left( \frac{2^p}{2} \right)^j \right) n \tilde{t}^p \) converges for \( p = 1/q < 1 \). It guarantees that, for an arbitrary given \( c > 0 \), there exists \( m_0 \geq 0 \) such that

\[
\sum_{j=m}^{m'+m-1} \left( \frac{1}{4} \left( \frac{2^p}{4} \right)^j + \frac{1}{6} \left( \frac{2^p}{2} \right)^j \right) n \tilde{t}^p < c
\]

for each \( m \geq m_0 \) and \( m' > 0 \). By (N5) and (8), we have

\[
N'(J_m f(x) - J_{m'+m} f(x), c) \\
\geq N' \left( J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \left( \frac{1}{4} \left( \frac{2^p}{4} \right)^j + \frac{1}{6} \left( \frac{2^p}{2} \right)^j \right) n \tilde{t}^p \right) \\
\geq N \left( x, \frac{\tilde{t}}{2} \right) \geq N(x, t_0) \geq 1 - \varepsilon
\]
for all $x \in X$. Hence $\{J_m f(x)\}$ is a Cauchy sequence in the fuzzy Banach space $(Y, N')$, and so we can define a mapping $F : X \to Y$ by

$$F(x) := N' - \lim_{m \to \infty} J_m f(x)$$

for all $x \in X$. Moreover, if we put $m = 0$ in (8), we have

$$N'(f(x) - J_m f(x), t) \geq N\left(x, \frac{t^q}{2 \left(\sum_{j=0}^{n'} 1 \left(\frac{2^p}{4}\right)^j + \frac{1}{6} \left(\frac{2^p}{2}\right) n\right)^q}\right)$$

(9)

for all $x \in X$. Next we will show that $F$ is a quadratic-additive mapping. Using (N4), we have

$$N'(DF(x_1, \ldots, x_n), t)$$

$$\geq \min \left\{ N\left(\sum_{i=1}^{n} (F - J_m f)\left(x_i - \sum_{j=1}^{n} \frac{x_j}{n}\right), \frac{t}{4n}\right) \right\},$$

$$\min \left\{ N'(F - J_m f)\left(x_i, \frac{t}{4n}\right), N'(DJ_m f(x_1, \ldots, x_n), \frac{t}{4}\right\}$$

(10)

for all $x_1, \ldots, x_n \in X$ and $m \in N$. The first three terms on the right hand side of (10) tend to 1 as $m \to \infty$ by the definition of $F$ and (N2), and the last term holds

$$N'(DJ_m f(x_1, x_2, \ldots, x_n), \frac{t}{4})$$

$$\geq \min \left\{ N'(Df(2^m x_1, \ldots, 2^m x_n), \frac{t}{16}), N'(Df(-2^m x_1, \ldots, -2^m x_n), \frac{t}{16}) \right\},$$

$$N'(Df(2^m x_1, \ldots, 2^m x_n), \frac{t}{16}), N'(Df(-2^m x_1, \ldots, -2^m x_n), \frac{t}{16}) \right\}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. By (N3) and (5), we obtain

$$N'(Df(\pm 2^m x_1, \ldots, \pm 2^m x_n), \frac{t}{16})$$

$$\geq \min \left\{ N\left(x_1, 2^{(2q-1)m-3q} n^{-q} t\right), \ldots, N\left(x_n, 2^{(2q-1)m-3q} n^{-q} t\right) \right\}$$

and

$$N'(Df(\pm 2^m x_1, \ldots, \pm 2^m x_n), \frac{t}{16})$$

$$\geq \min \left\{ N\left(x_1, 2^{(q-1)m-3q} n^{-q} t\right), \ldots, N\left(x_n, 2^{(q-1)m-3q} n^{-q} t\right) \right\}$$
for all $x_1, \ldots, x_n \in X$ and $m \in \mathbb{N}$. Since $q > 1$, together with (N5), we can deduce that the last term of (10) also tends to 1 as $m \to \infty$. It follows from (10) that

$$N'(DF(x_1, x_2, \ldots, x_n), t) = 1$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. By (N2), we have $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$. Now we approximate the difference between $f$ and $F$ in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since $F$ is the limit of $\{J_m(f)\}$, there is $m \in \mathbb{N}$ such that

$$N'(F(x) - J_m f(x), t - t') \geq 1 - \varepsilon.$$

By (9), we have

$$N'(F(x) - f(x), t) \geq \min \left\{ N'(F(x) - J_m f(x), t - t') , N'(J_m f(x) - f(x), t') \right\}$$

$$\geq \min \left\{ 1 - \varepsilon, N \left( x, \frac{t^q}{2 \left( \sum_{j=0}^{m-1} \left( \frac{1}{4} \left( \frac{2^j}{4} \right)^j \frac{1}{4} \left( \frac{2^j}{4} \right)^j \left( \frac{n}{2-2^j} \right) q \right) \right) \right) \right\}$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (6) in this case. Finally, to prove the uniqueness of $F$, let $F' : X \to Y$ be another quadratic-additive mapping satisfying (6). Since $F' : X \to Y$ is a quadratic-additive mapping, $F'$ satisfies the equality

$$F'(x) = J_m F'(x)$$

for all $x \in X$ and $m \in \mathbb{N}$. Together with (N4) and (6), this implies that

$$\begin{align*}
N'(F'(x) - J_m f(x), t) & = N'(J_m F'(x) - J_m f(x), t) \\
& \geq \min \left\{ N' \left( \frac{(F' - f)(2^m x)}{2 \cdot 4^m} , t \right) , N' \left( \frac{(F' - f)(-2^m x)}{2 \cdot 4^m} , t \right) \right\} \\
& \geq \sup_{t' < t} N \left( x, 2^{(q-1)m-q} \left( \frac{t'^q}{2 \left( \frac{n}{4-2^j} + \frac{n}{3(2-2^j)} \right) q} \right) \right)
\end{align*}$$

for all $x \in X$ and $m \in \mathbb{N}$. Observe that, for $q = \frac{1}{p}$, the last term of the above inequality tends to 1 as $m \to \infty$ by (N5). $F'(x) = N' - \lim_{m \to \infty} J_m f(x)$ and so we get

$$F(x) = F'(x)$$
for all \( x \in X \).

**Case 2.** Let \( \frac{1}{2} < q < 1 \) and let \( J_{m}f : X \to Y \) be a mapping defined by

\[
J_{m}f(x) := 2^{-2m-1}(f(2^{m}x) + f(-2^{m}x)) + 2^{m-1}\left(f\left(\frac{x}{2^{m}}\right) - f\left(-\frac{x}{2^{m}}\right)\right)
\]

for all \( x \in X \). Then we have \( J_{0}f(x) = f(x) \) and

\[
J_{j}f(x) - J_{j+1}f(x) = \frac{Df(\Delta(2^{j}x))}{2 \cdot 4^{j+1}} + \frac{Df(\Delta(-2^{j}x))}{2 \cdot 4^{j+1}}
\]

\[
- \frac{2^{j-1}Df(\Delta(x_{2j+1}))}{3} + \frac{2^{j-1}Df(\Delta(-x_{2j+1}))}{3}
\]

for all \( x \in X \) and \( j \geq 0 \). If \( m' + m > m \geq 0 \), then

\[
N'(J_{m}f(x) - J_{m'+m}f(x), \sum_{j=m}^{m'+m-1} \left(\frac{1}{4} \left(\frac{2^{p}}{4}\right)^{j} + \frac{1}{3 \cdot 2^{p}} \left(\frac{2}{2^{p}}\right)^{j}\right) nt^{p})
\]

\[
\geq N'\left(\sum_{j=m}^{m'+m-1} (J_{j}f(x) - J_{j+1}f(x)), \sum_{j=m}^{n+m-1} \left(\frac{1}{4} \left(\frac{2^{p}}{4}\right)^{j} + \frac{1}{3 \cdot 2^{p}} \left(\frac{2}{2^{p}}\right)^{j}\right) nt^{p}\right)
\]

\[
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N'\left(J_{j}f(x) - J_{j+1}f(x), \left(\frac{1}{4} \left(\frac{2^{p}}{4}\right)^{j} + \frac{1}{3 \cdot 2^{p}} \left(\frac{2}{2^{p}}\right)^{j}\right) nt^{p}\right)\right\}
\]

\[
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ \min \left\{ N'\left(\frac{Df(\Delta(2^{j}x))}{2 \cdot 4^{j+1}}, \frac{2^{j}nt^{p}}{2 \cdot 4^{j+1}}\right), N'\left(\frac{Df(\Delta(-2^{j}x))}{2 \cdot 4^{j+1}}, \frac{2^{j}nt^{p}}{2 \cdot 4^{j+1}}\right), N'\left(\frac{2^{j-1}Df(\Delta(x_{2j+1}))}{3}, \frac{2^{j-1}nt^{p}}{3 \cdot 2^{j+1}}\right)\right\}\right\}
\]

\[
\geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N(2^{j}x, 2^{j}t), N(2^{j+1}x, 2^{j}t), N\left(\frac{x}{2^{j+1}}, \frac{t}{2^{j+1}}\right), N\left(\frac{x}{2^{j}}, \frac{t}{2^{j+1}}\right)\right\}
\]

\[
= N\left(\frac{x}{2}, \frac{t}{2}\right)
\]

for all \( x \in X \) and \( t > 0 \). In the similar argument following (6) of the previous case, we can define the limit \( F(x) := N' - \lim_{m \to \infty} J_{m}f(x) \) of the Cauchy sequence \( \{J_{m}f(x)\} \) in the Banach fuzzy space \( Y \). Moreover, putting \( m = 0 \) in the above inequality, we have

\[
N'(f(x) - J_{m'}f(x), t) \geq N\left(x, \frac{t^{q}}{2 \left(\sum_{j=0}^{m'-1} \left(\frac{1}{4} \left(\frac{2^{p}}{4}\right)^{j} + \frac{1}{3 \cdot 2^{p}} \left(\frac{2}{2^{p}}\right)^{j}\right) n^{q}\right)\right)
\]
for each \( x \in X \). To prove that \( F \) is a quadratic additive mapping, we have enough to show that the last term of (10) in Case 1 tends to 1 as \( m \to \infty \). By (N3) and (5), we get

\[
N'(DJ_m f(x_1, x_2, \ldots, x_n), \frac{t}{4}) \\
\geq \min \left\{ N' \left( \frac{Df (2^m x_1, \ldots, 2^m x_n)}{2 \cdot 4^m}, \frac{t}{16} \right), N' \left( \frac{Df (-2^m x_1, \ldots, -2^m x_n)}{2 \cdot 4^m}, \frac{t}{16} \right), N' \left( \frac{2^{m-1} Df (\frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m})}{2 \cdot 4^m}, \frac{t}{16} \right), N' \left( \frac{-2^{m-1} Df (-\frac{x_1}{2^m}, \ldots, -\frac{x_n}{2^m})}{2 \cdot 4^m}, \frac{t}{16} \right) \right\}
\]

for all \( x_1, x_2, \ldots, x_n \in X \) and \( t > 0 \). Observe that all the terms on the right hand side of the above inequality tend to 1 as \( m \to \infty \), since \( \frac{1}{2} < q < 1 \). Hence, together with the similar argument after (10), we can say that \( DF(x_1, x_2, \ldots, x_n) = 0 \) for all \( x_1, x_2, \ldots, x_n \in X \). Recall, in Case 1, the inequality (6) follows from (9). By the same reasoning, we get (6) from (11) in this case. Now to prove the uniqueness of \( F \), let \( F' \) be another quadratic additive mapping satisfying (6). Then, together with (N4) and (6), we have

\[
N'(F'(x) - J_m f(x), t) \\
= N'(J_m F'(x) - J_m f(x), t) \\
\geq \min \left\{ N' \left( \frac{(F' - f)(2^m x)}{2 \cdot 4^m}, \frac{t}{4} \right), N' \left( \frac{(F' - f)(-2^m x)}{2 \cdot 4^m}, \frac{t}{4} \right), N' \left( \frac{2^{m-1}(F' - f)(\frac{x}{2^m})}{2 \cdot 4^m}, \frac{t}{4} \right), N' \left( \frac{-2^{m-1}(F' - f)(-\frac{x}{2^m})}{2 \cdot 4^m}, \frac{t}{4} \right) \right\} \\
\geq \min \left\{ \sup_{t < t} N \left( x, \frac{2(2q-1)m-q t^q}{2(\frac{n}{4} - 2^{q-1} m + \frac{n}{3(2^{q-1} m - q)} t^q)} \right), \sup_{t < t} N \left( x, \frac{2(1-q)m-q t^q}{2(\frac{n}{4} - 2^{q-1} m + \frac{n}{3(2^{q-1} m - q)} t^q)} \right) \right\}
\]

for all \( x \in X \) and \( m \in \mathbb{N} \). Since \( \lim_{m \to \infty} 2^{(2q-1)m-q} = \lim_{m \to \infty} 2^{(1-q)m-q} = \infty \) in this case, both terms on the right hand side of the above inequality tend to 1 as \( m \to \infty \) by (N5). This implies that \( F'(x) = N' - \lim_{m \to \infty} J_m f(x) \) and so \( F(x) = F'(x) \) for all \( x \in X \).

**Case 3.** Finally, we take \( 0 < q < \frac{1}{2} \) and define \( J_m f : X \to Y \) by

\[
J_m f(x) := \frac{1}{2} \left( 4^m (f(2^{-m} x) + f(-2^{-m} x)) + 2^m (f(2^{-m} x) - f(-2^{-m} x)) \right)
\]
for all $x \in X$. Then we have $J_0f(x) = f(x)$ and
\[
J_jf(x) - J_{j+1}f(x) = -2 \cdot 4^j Df \left( \Delta \left( \frac{x}{2^{j+1}} \right) \right) - 2 \cdot 4^{j+1} Df \left( \Delta \left( -\frac{x}{2^{j+1}} \right) \right)
\]
which implies that if $m' + m > m \geq 0$ then
\[
N' \left( J_m f(x) - J_{m'} f(x), \sum_{j=m}^{m'+m-1} \frac{1}{2^m} \left( \frac{4}{2^p} \right)^j nt^p \right)
\]
\[
\geq \min \left\{ \sum_{j=m}^{m'+m-1} \min \left\{ N' \left( \frac{3 \cdot 4^j + 2j}{6} Df \left( \Delta \left( \frac{x}{2^j+1} \right) \right), \frac{3 \cdot 4^j + 2j}{6} nt^p \right) \right\}, \left( x, \frac{t}{2^j+1} \right), N' \left( \frac{x}{2^j+1}, \frac{t}{2^j+1} \right) \right\}
\]
\[
= N \left( x, \frac{t}{2^j+1} \right)
\]
for all $x \in X$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \to Y$ by $F(x) := N' - \lim_{m \to \infty} J_m f(x)$. Putting $m = 0$ in the above inequality, we have
\[
N'(f(x) - J_m f(x), t) \geq N \left( x, \frac{t^q}{2 \left( n \sum_{j=0}^{m'-1} \frac{1}{2^p} \left( \frac{4}{2^p} \right)^j \right)^q} \right)
\]  
(12)
for all $x \in X$ and $t > 0$. Notice that
\[
N' \left( DJ_m f(x_1, x_2, \ldots, x_n), \frac{t}{4} \right)
\]
\[
\geq \min \left\{ N' \left( 4^m Df \left( \frac{x_1}{2}, \ldots, \frac{x_n}{2m} \right), \frac{t}{16} \right), N' \left( 4^m Df \left( -\frac{x_1}{2}, \ldots, -\frac{x_n}{2m} \right), \frac{t}{16} \right), \right. \right.
\]
\]
\[
N' \left( 2^{m-1} Df \left( \frac{x_1}{2^m}, \ldots, \frac{x_n}{2^m} \right), \frac{t}{16} \right), \right. \right.
\]
\[
N' \left( -2^{m-1} Df \left( -\frac{x_1}{2^m}, \ldots, -\frac{x_n}{2^m} \right), \frac{t}{16} \right) \right\}
\]
\[
\geq \min \left\{ N \left( x_1, 2^{1-2q} m^{-3q} n^{-1} t^q \right), \ldots, N \left( x_n, 2^{1-2q} m^{-3q} n^{-1} t^q \right), \right. \right.
\]
\[
N \left( x_1, 2^{1-q} m^{-3q} n^{-1} t^q \right), \ldots, N \left( x_n, 2^{1-q} m^{-3q} n^{-1} t^q \right) \right\}
for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. Since $0 < q < \frac{1}{2}$, all terms on the right hand side tend to 1 as $m \to \infty$, which implies that the last term of (10) tends to 1 as $m \to \infty$. Therefore, we can say that $DF \equiv 0$. Moreover, using the similar argument after (10) in Case 1, we get the inequality (6) from (12) in this case.

To prove the uniqueness of $F$, let $F' : X \to Y$ be another quadratic additive function satisfying (6). Then together with (N4) and (6), we get

$$N'(F'(x) - J_m f(x), t) \geq \min \left\{ N' \left( \frac{4^m}{2} (F' - f) \left( \frac{x}{2^m} \right), \frac{t}{4} \right), N' \left( \frac{4^m}{2} (F' - f) \left( \frac{-x}{2^m} \right), \frac{t}{4} \right), N' \left( 2^{m-1} (F' - f) \left( \frac{x}{2^m} \right), \frac{t}{4} \right), N' \left( 2^{m-1} (F' - f) \left( \frac{-x}{2^m} \right), \frac{t}{4} \right) \right\} \geq \sup_{t' < t} N \left( x, \frac{2(1-2q)m-q(2p-4)pt'}{2 \cdot n^q} \right)$$

for all $x \in X$ and $m \in \mathbb{N}$. Since $q < \frac{1}{2}$, the last term tends to 1 as $m \to \infty$ by (N5). This implies that $F'(x) = N' - \lim_{m \to \infty} J_m f(x)$ and so $F(x) = F'(x)$ for all $x \in X$. It completes the proof of Theorem 2.2.

We can use Theorem 2.2 to get a classical result in the framework of normed spaces. Let $(X, \| \cdot \|)$ be a normed linear space. Suppose that $f : X \to Y$ is a mapping into a Banach space $(Y, ||| \cdot |||)$ such that

$$|||Df(x_1, x_2, \ldots, x_n)||| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$$

for all $x_1, x_2, \ldots, x_n \in X$, where $p > 0$ and $p \neq 1, 2$. Then the inequality

$$N_Y(Df(x_1, \ldots, x_n, t_1 + \cdots + t_n) \geq \min \{N_X(x_1, t_1^q), \ldots, N_X(x_n, t_n^q)\}$$

holds for $q = \frac{1}{p}$, where $N_X$ and $N_Y$ are fuzzy norms defined by

$$N_X(x, t) = \begin{cases} 0, & t \leq \|x\|, \\ 1, & t > \|x\| \end{cases} \quad \text{and} \quad N_Y(y, t) = \begin{cases} 0, & t \leq |||y|||, \\ 1, & t > |||y||| \end{cases}$$

for all $x \in X$, $y \in Y$, and $t \in \mathbb{R}$, see [18]. It means that $f$ is a fuzzy $q$-almost quadratic additive mapping, and by Theorem 2.2, we get the following stability result.

**Corollary 2.3** Let $(X, \| \cdot \|)$ be a normed linear space and let $(Y, ||| \cdot |||)$ be a Banach space. If $n$ is a natural number greater than 2 and $f : X \to Y$ satisfies

$$|||Df(x_1, x_2, \ldots, x_n)||| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$$
for all \( x_1, x_2, \ldots, x_n \in X \), where \( p > 0 \) and \( p \neq 1, 2 \), then there is a unique quadratic-additive mapping \( F : X \to Y \) such that

\[
\| F(x) - f(x) \| \leq \begin{cases} 
\left( \frac{2n}{|2p-4|} + \frac{2n}{3|2p-2|} \right) \| x \|^p & \text{if } 0 < p < 2 \text{ and } p \neq 1, \\
\frac{2p-4}{2p} & \text{if } p > 2
\end{cases}
\]

for all \( x \in X \).

**Remark 2.4** Consider a mapping \( f : X \to Y \) satisfying (5) for a real number \( q < 0 \). Take any \( t > 0 \). If we choose a real number \( s \) with \( 0 < ns < t \), then we have

\[ N'(Df(x_1, \ldots, x_n), t) \geq N'(Df(x_1, \ldots, x_n), ns) \geq \min \{N(x_1, s^q), \ldots, N(x_n, s^q)\} \]

for all \( x_1, x_2, \ldots, x_n \in X \). Since \( q < 0 \), we have \( \lim_{s \to 0^+} s^q = \infty \). This implies that

\[ \lim_{s \to 0^+} N(x_1, s^q) = \cdots = \lim_{s \to 0^+} N(x_n, s^q) = 1 \]

and so

\[ N'(Df(x_1, \ldots, x_n), t) = 1 \]

for all \( x_1, \ldots, x_n \in X \) and \( t > 0 \). By (N2), it allows us to get \( Df(x_1, \ldots, x_n) = 0 \) for all \( x_1, \ldots, x_n \in X \). In other words, \( f \) is itself a quadratic additive mapping if \( f \) is a fuzzy \( q \)-almost quadratic-additive mapping for the case \( q < 0 \).

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**References**


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