A Fixed Point Approach to the Stability of a Quadratic-Additive Type Functional Equation in Non-Archimedean Normed Spaces

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Abstract

In this paper, we investigate the generalized Hyers–Ulam stability for the functional equation

\[ f(ax + y) + af(y - x) - \frac{a(a + 1)}{2} f(x) - \frac{a(a + 1)}{2} f(-x) - (a + 1) f(y) = 0 \]

in non-Archimedean normed spaces.

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1 Introduction

A stability problem of functional equations was formulated by S. M. Ulam [14] in 1940, as follows: “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?”. In following year, D. H. Hyers [5] gave a partial solution of Ulam’s problem for the case of approximate additive mappings between Banach spaces. Hyers’ Theorem was generalized by T. Aoki [1] and D. G. Bourgin [2] for additive mappings by allowing the Cauchy difference operator to be controlled by $\varepsilon(||x||^p + ||y||^p)$. In 1978, Th. M. Rassias [12] also provided a generalization of Hyers’ Theorem for linear mappings by allowing the Cauchy difference operator to be controlled by $\varepsilon(||x||^p + ||y||^p)$, which was generalized by P. Gavruta in [4] and S.-M. Jung in [6].

By a non-Archimedean field we mean a field $K$ equipped with a function $(valuation)$ $|·|$ from $K$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |−1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. Let $X$ be a vector space over a scalar field $K$ with a non-Archimedean non-trivial valuation $|·|$. A function $||·|| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$;
(ii) $||rx|| = |r||x||$ ($r \in K, x \in X$);
(iii) the strong triangle inequality (ultrametric); namely,

$$||x + y|| \leq \max\{||x||, ||y||\} \quad (x, y \in X).$$

Then $(X, ||·||)$ is called a non-Archimedean space. Due to the fact that

$$||x_n - x_m|| \leq \max\{||x_{j+1} - x_j|| : m \leq j \leq n - 1\}, \quad (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent in the space. Recently, M. S. Moslehian and Th. M. Rassias [11] discussed the Hyers–Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean normed spaces.

Now we consider the quadratic-additive type functional equation

$$f(ax + y) + af(y - x) - \frac{a(a + 1)}{2}f(x) - \frac{a(a + 1)}{2}f(-x) - (a + 1)f(y) = 0 \quad (1)$$

whose solution is called a quadratic-additive mapping. For the case $a = 1$, Lee [10], H.-K. Kim and Y.-H. Lee [9], and G.-H. Kim and Y.-H. Lee [8] investigated the stability of the equation (1) on restricted domains, in Fuzzy spaces, and in nonarchimedean spaces, respectively.

In this paper, we are going to investigate alternative stability results of the quadratic-additive functional equation (1) in non-Archimedean normed spaces by using the fixed point method.
2 Generalized Hyers-Ulam stability of (1)

We recall the following result of the fixed point theory by Diaz and Margolis.

**Theorem 2.1** ([3, 13]) Suppose that a complete generalized metric space $(X,d)$, which means that the metric $d$ may assume infinite value, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = +\infty$$

for all positive integers $n$, or there exists a nonnegative integer $k$ such that:

1. $d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;
2. the sequence $\{J^n x\}$ is convergent to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in $Z := \{y \in X : d(J^k x, y) < +\infty\}$;
4. $d(y, y^*) \leq (1/(1 - L))d(y, Jy)$ for all $y \in Z$.

Let $V$ and $W$ be vector spaces. For a given mapping $f : V \to W$, we use the abbreviations

$$Af(x, y) := f(x + y) - f(x) - f(y),$$
$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y),$$
$$Da f(x, y) := f(ax + y) + af(y - x) - \frac{a(a + 1)}{2} f(x)$$
$$- \frac{a(a + 1)}{2} f(-x) - (a + 1)f(y)$$

for all $x, y \in V$, where $a$ is a fixed positive integer. We call a mapping $f$ an additive mapping or a quadratic mapping if $Af(x) = 0$ or $Qf(x) = 0$ for all $x, y \in V$, respectively.

**Theorem 2.2** Let $V$ and $W$ be vector spaces. A mapping $f : V \to W$ satisfies the inequality $Da f(x, y) = 0$ for all $x, y \in V$ if and only if there exist a quadratic mapping $g : V \to W$ and an additive mapping $h : V \to W$ such that

$$f(x) = g(x) + h(x)$$

for all $x \in V$. 
Proof (Necessity) We decompose \( f \) into the even part and the odd part by putting

\[
g(x) = \frac{f(x) + f(-x)}{2}, \quad h(x) = \frac{f(x) - f(-x)}{2}
\]

for all \( x \in V \). Notice that \( f(0) = -\frac{D_a f(0,0)}{a(a+1)} = 0 \). From the equalities

\[
Qg(x,y) = \frac{D_a g(y, (a+1)x) - D_a g(x+y, x)}{a(a+1)} - \frac{D_a g(-x, y-x)}{a+1} + \frac{D_a g(x, x)}{a}
\]

\[
Ah(x,y) = \frac{D_a h(y, ax) + (a+1)D_a h(x, 0) - D_a h(x+y, 0)}{a(a+1)} + \frac{D_a h(-x, y)}{a+1}
\]

for all \( x, y \in V \), we conclude that \( g \) is a quadratic mapping and \( h \) is an additive mapping.

(Sufficiency) If there exist a quadratic mapping \( g : V \to W \) and an additive mapping \( h : V \to W \) such that \( f(x) = g(x) + h(x) \) for all \( x \in V \), then \( g(0) = h(0) = 0 \), \( g(-x) = g(x) \), and

\[
D_1 g(x,y) = Qg(x,y) = 0
\]

for all \( x, y \in V \). Assume that \( D_k g(x,y) = 0 \) for all \( x, y \in V \) and \( k \geq 1 \). Then

\[
D_{k+1} g(x,y) = D_k g(x,x+y) + (k+1)Qg(x,y) = 0
\]

for all \( x, y \in V \). By induction, it follows that

\[
D_a g(x,y) = 0
\]

for all \( x, y \in V \). Since \( h \) is an additive mapping, we have \( h(-x) = -h(x) \), \( h(ax) = ah(x) \), and

\[
D_a h(x,y) = Ah(ax,y) + aAh(y,-x) = 0
\]

for all \( x, y \in V \).

Now we prove the generalized Hyers-Ulam stability of the functional equation \( Df \equiv 0 \), using Theorem 2.1.

**Theorem 2.3** Let \( V \) be a vector space over a scalar field \( K \), and let \((Y, \| \cdot \|)\) be a complete non-Archimedean space over \( K \). Let \( \varphi : V^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[
\varphi(x,y) \leq 2\| (a+1)^2 |L \varphi \left( \frac{x}{a+1}, \frac{y}{a+1} \right) \|
\]

(2)
for all \( x, y \in V \). Suppose that \( f : V \to Y \) is a mapping satisfying
\[
\| D_a f(x, y) \| \leq \varphi(x, y)
\]
for all \( x, y \in V \). Then there exists a unique quadratic-additive mapping \( T : V \to Y \) such that
\[
\| f(x) - T(x) \| \leq \frac{\tilde{\varphi}(x)}{2(a+1)^2|1-L|}
\]
for all \( x \in V \), where \( \tilde{\varphi} \) is given by
\[
\tilde{\varphi}(x) := \max \{ \varphi(x, x), \varphi(-x, -x) \}
\]
for all \( x \in V \). In particular the mapping \( T \) is represented by
\[
T(x) = \lim_{n \to \infty} \left[ f((a+1)x) + f(-((a+1)x)) + f(-((a+1)x)) - f(-(a+1)x) - f((a+1)x) - f(-((a+1)x)) \right]
\]
for all \( x \in V \).

**Proof.** Let \( S \) be the set of all functions \( g : V \to Y \) with \( g(0) = 0 \). We introduce a generalized metric on \( S \) by
\[
d(g, h) := \inf \{ K \in [0, \infty] | \| g(x) - h(x) \| \leq K \tilde{\varphi}(x) \text{ for all } x \in V \}.
\]
It is not difficult to show that \( (S, d) \) is a complete generalized metric space (see [7]). Now we consider the operator \( J : S \to S \) defined by
\[
Jg(x) := \frac{g((a+1)x) - g(-(a+1)x)}{2(a+1)} + \frac{g((a+1)x) + g(-(a+1)x)}{2(a+1)^2}
\]
for all \( x \in V \). Let \( g, h \in S \) be given such that \( d(g, h) < \beta \); by the definition,
\[
\| g(x) - h(x) \| \leq \beta \tilde{\varphi}(x)
\]
for all \( x \in V \). Hence, we have
\[
\| Jg(x) - Jh(x) \| \leq \max \left\{ \left\| \frac{(a+2)(g((a+1)x) - h((a+1)x))}{2(a+1)^2} \right\|, \left\| \frac{ag(-(a+1)x) - h(-(a+1)x)}{2(a+1)^2} \right\| \right\}
\]
\[
\leq \beta \max \left\{ \left| \frac{a+2}{2(a+1)^2} \right| \tilde{\varphi}(x), \left| \frac{a}{2(a+1)^2} \right| \tilde{\varphi}(-x) \right\}
\]
\[
\leq \beta \max \left\{ \tilde{\varphi}(x), \tilde{\varphi}(-x) \right\}
\]
\[
\leq \beta L \max \left\{ (\tilde{\varphi}(x), \tilde{\varphi}(-x)) \right\}
\]
for all \( x \in V \), which implies that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for any \( g, h \in S \). That is, \( J \) is a strict contraction with the Lipschitz constant \( L \). Moreover, by (2), we see that
\[
\|f(x) - Jf(x)\| = \left\| \frac{a}{2(a + 1)^2} D_a f(-x, -x) - \frac{a + 2}{2(a + 1)^2} D_a f(x, x) \right\|
\]
\[
\leq \max \left\{ \frac{|a + 2| \varphi(x, x)}{|2(a + 1)^2|}, \frac{|a| \varphi(-x, -x)}{|2(a + 1)^2|} \right\}
\]
\[
\leq \frac{1}{|2(a + 1)^2|} \max \{ \varphi(x, x), \varphi(-x, -x) \}
\]
for all \( x \in V \), i.e., \( d(f, Jf) \leq \frac{1}{|2(a + 1)^2|} < \infty \) (see the definition of \( d \)). Therefore, by Theorem 2.1 (2), the sequence \( \{J^n\} \) is convergent to a fixed point \( T \) of \( J \), i.e.,
\[
T(x) := \lim_{n \to \infty} J^n f(x)
\]
\[
= \lim_{n \to \infty} \left[ \frac{f((a + 1)^n x + f(-(a + 1)^n x) + f((a + 1)^n x) + f(-(a + 1)^n x)}{2(a + 1)^n} \right]
\]
for all \( x \in V \). Moreover, we have
\[
d(f, T) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{1}{|2(a + 1)^2|(1 - L)}
\]
which implies the validity of (4). Replacing \( x, y \) by \( (a + 1)^n x, (a + 1)^n y \), respectively, in (2) and (3), we have
\[
\|D_a T(x, y)\|
\]
\[
= \lim_{n \to \infty} \left\| D_a f((a + 1)^n x, (a + 1)^n y) - D_a f(-(a + 1)^n x, -(a + 1)^n y) \right\|
\]
\[
\leq \lim_{n \to \infty} \max \left\{ \frac{|(a + 1)^n + 1|}{|2(a + 1)^{2n}|} \varphi((a + 1)^n x, (a + 1)^n y), \frac{|(a + 1)^n + 1|}{|2(a + 1)^{2n}|} \varphi(-(a + 1)^n x, -(a + 1)^n y) \right\}
\]
\[
\leq \lim_{n \to \infty} \max \left\{ \frac{1}{|2(a + 1)^{2n}|} \varphi((a + 1)^n x, (a + 1)^n y), \frac{1}{|2(a + 1)^{2n}|} \varphi(-(a + 1)^n x, -(a + 1)^n y) \right\}
\]
\[
= \lim_{n \to \infty} L^n \max \{ \varphi(x, y), \varphi(-x, -y) \}
\]
\[
= 0
\]
Stability of quadratic-additive type functional equations

1483

for all \( x, y \in V \). This follows that \( T \) is a quadratic-additive mapping. To prove the uniqueness assertion, let us assume that there exists another quadratic-additive mapping \( T' : V \to Y \) which satisfies (4). Then, by Theorem 2.2, we get \( QT'_e \equiv 0 \) and \( AT'_o \equiv 0 \) which follows that

\[ T'(x) = T'_e(x) + T'_o(x) = \frac{T'_e((a + 1)x)}{(a + 1)^2} + \frac{T'_o((a + 1)x)}{(a + 1)} = JT'(x) \]

for all \( x \in V \), i.e., \( T' \) is a fixed point of \( J \) in \( S \). However, by Theorem 2.1, \( J \) has only one fixed point in \( S \), and hence \( T = T' \). This completes the proof.

**Theorem 2.4** Let \( V \) be a vector space over a scalar field \( K \), and let \((Y, \| \cdot \|)\) be a complete non-Archimedean space over \( K \). Let \( \varphi : V^2 \to [0, \infty) \) be a function such that there exists an \( L < 1 \) with

\[ \varphi(x, y) \leq |2\frac{L}{a + 1}| \varphi((a + 1)x, (a + 1)y) \]  

(5)

for all \( x, y \in V \). Suppose that \( f : V \to Y \) is a mapping satisfying

\[ \|D_a f(x, y)\| \leq \varphi(x, y) \]  

(6)

for all \( x, y \in V \). Then there exists a unique quadratic-additive mapping \( T : V \to Y \) such that

\[ \|f(x) - T(x)\| \leq \frac{|2|L\tilde{\varphi}(x)}{|a + 1|(1 - L)} \]  

(7)

for all \( x \in V \), where \( \tilde{\varphi} \) is given by

\[ \tilde{\varphi}(x) := \max\{\varphi(x, x), \varphi(-x, -x)\} \]

for all \( x \in V \). In particular the mapping \( T \) is represented by

\[ T(x) = \lim_{n \to \infty} \left[ \frac{(a + 1)^{2n}}{2} \left( f\left( \frac{x}{(a + 1)^n} \right) + f\left( \frac{-x}{(a + 1)^n} \right) \right) + \frac{(a + 1)^n}{2} \left( f\left( \frac{x}{(a + 1)^n} \right) - f\left( \frac{-x}{(a + 1)^n} \right) \right) \right] \]

for all \( x \in V \).

**Proof.** Let \( S \) be the set of all functions \( g : V \to Y \) with \( g(0) = 0 \). We introduce a generalized metric on \( S \) by

\[ d(g, h) := \inf \{K \in [0, \infty] \mid \|g(x) - h(x)\| \leq K\tilde{\varphi}(x) \text{ for all } x \in V \}. \]
It is not difficult to show that \((S,d)\) is a complete generalized metric space (see [7]).

Now we consider the operator \(J : S \rightarrow S\) defined by
\[
Jg(x) := \frac{(a + 1)^2}{2} \left( g\left( \frac{x}{a + 1}\right) + g\left( -\frac{x}{a + 1}\right) \right) + \frac{a + 1}{2} \left( g\left( \frac{x}{a + 1}\right) - g\left( -\frac{x}{a + 1}\right) \right)
\]
for all \(x \in V\). Let \(g, h \in S\) be given such that \(d(g, h) < \beta\); by the definition,
\[
\|g(x) - h(x)\| \leq \beta \tilde{\varphi}(x)
\]
for all \(x \in V\). Hence, we have
\[
\|Jg(x) - Jh(x)\| \\
\leq \beta \max \left\{ \left\| \frac{(a + 1)^2 + (a + 1)}{2} \left( g\left( \frac{x}{a + 1}\right) - h\left( \frac{x}{a + 1}\right) \right) \right\|, \right. \\
\left. \left\| \frac{(a + 1)^2 - a - 1}{2} \left( g\left( \frac{-x}{a + 1}\right) - h\left( \frac{-x}{a + 1}\right) \right) \right\| \right\}
\leq \beta \max \left\{ \frac{|(a + 1)^2 + a + 1|}{|2|} \tilde{\varphi}\left( \frac{x}{a + 1}\right), \frac{|(a + 1)^2 - a - 1|}{|2|} \tilde{\varphi}\left( \frac{-x}{a + 1}\right) \right\}
\leq \beta \max \left\{ \frac{|a + 1|}{|2|} \tilde{\varphi}\left( \frac{x}{a + 1}\right), \frac{|a + 1|}{|2|} \tilde{\varphi}\left( \frac{-x}{a + 1}\right) \right\}
\leq \beta L \max \{ (\tilde{\varphi}(x), \tilde{\varphi}(-x)) \}
\]
for all \(x \in V\), which implies that
\[
d(Jg, Jh) \leq Ld(g, h)
\]
for any \(g, h \in S\). That is, \(J\) is a strict contraction with the Lipschitz constant \(L\). Moreover, by (5), we see that
\[
\|f(x) - Jf(x)\| = \left\| D_a f\left( \frac{x}{a + 1}, \frac{x}{a + 1}\right) \right\|
\leq \varphi\left( \frac{x}{a + 1}, \frac{x}{a + 1}\right)
\leq \frac{|2L|}{|a + 1|} \max \{ \varphi(x, x), \varphi(-x, -x) \}
\]
for all \(x \in V\), i.e., \(d(f, Jf) \leq \frac{|2L|}{|a + 1|} < \infty\) (see the definition of \(d\)). Therefore, by Theorem 2.1 (2), the sequence \(\{J^n\}\) is convergent to a fixed point \(T\) of \(J\), i.e.,
\[
T(x) := \lim_{n \to \infty} J^n f(x) = \lim_{n \to \infty} \left[ \left( \frac{a + 1}{2} \right)^n \left( f\left( \frac{x}{(a + 1)^n}\right) + f\left( -\frac{x}{(a + 1)^n}\right) \right) \right]
\leq \frac{(a + 1)^n}{2} \left( f\left( \frac{x}{(a + 1)^n}\right) + f\left( -\frac{x}{(a + 1)^n}\right) \right) + \frac{(a + 1)^n}{2} \left( f\left( \frac{x}{(a + 1)^n}\right) - f\left( \frac{-x}{(a + 1)^n}\right) \right).
\]
Moreover, we have
\[ d(f, T) \leq \frac{1}{1-L}d(f, Jf) \leq \frac{|2L|}{|a+1|(1-L)} \]
which implies the validity of (7). Replacing \( x, y \) by \( \frac{x}{(a+1)^n}, \frac{y}{(a+1)^n} \), respectively, in (5) and (6), we have
\[
\| D_a T(x, y) \| = \lim_{n \to \infty} \left\| \frac{(a+1)^n}{2} \left( D_a f \left( \frac{x}{(a+1)^n}, \frac{y}{(a+1)^n} \right) 
- D_a f \left( -\frac{x}{(a+1)^n}, -\frac{y}{(a+1)^n} \right) + \frac{(a+1)^{2n}}{2} \left( D_a f \left( \frac{x}{(a+1)^n}, \frac{y}{(a+1)^n} \right) 
+ D_a f \left( -\frac{x}{(a+1)^n}, -\frac{y}{(a+1)^n} \right) \right) \right\|
\leq \lim_{n \to \infty} \max \left\{ \frac{|(a+1)^{2n} + (a+1)^n|}{2}, \frac{|(a+1)^{2n} - (a+1)^n|}{2} \right\} \varphi \left( \frac{x}{(a+1)^n}, \frac{y}{(a+1)^n} \right),
\leq \lim_{n \to \infty} \max \left\{ \frac{|(a+1)^n|}{2}, \frac{|(a+1)^n|}{2} \right\} \varphi \left( \frac{x}{(a+1)^n}, \frac{y}{(a+1)^n} \right),
\leq \lim_{n \to \infty} L^n \max \{ \varphi(x, y), \varphi(-x, -y) \}
= 0
\]
for all \( x, y \in V \). To prove the uniqueness assertion, let us assume that there exists a quadratic-additive mapping \( T' : X \to Y \) which satisfies (3.4). Then, by Theorem 2.2, we get \( QT' = 0 \) and \( AT_0 = 0 \) which follows that
\[ T'(x) = T'_e(x) + T'_o(x) = (a+1)^2T'_e \left( \frac{x}{a+1} \right) + (a+1)T'_o \left( \frac{x}{a+1} \right) = JT'(x) \]
for all \( x \in V \), i.e., \( T' \) is a fixed point of \( J \) in \( S \). However, by Theorem 2.1, \( J \) has only one fixed point in \( S \), and hence \( T = T' \). This completes the proof.

References

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