A Method to Solve Non-homogeneous Strongly Coupled Mixed Parabolic Boundary Value Systems with Non-homogeneous Boundary Conditions

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Abstract

In this paper, a method to construct the solution of non-homogeneous parabolic coupled systems with non-homogeneous boundary conditions of the type $u_t - Au_{xx} = G(x, t)$, $A_1 u(0, t) + B_1 u_x(0, t) = P(t)$, $A_2 u(l, t) + B_2 u_x(l, t) = Q(t)$, $0 < x < 1$, $t > 0$, $u(x, 0) = f(x)$, where $A$ is a positive stable matrix and $A_1, A_2, B_1, B_2$ are arbitrary matrices for which the block matrix $\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix}$ is non-singular, is proposed. Two illustrative examples of the method are given.
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1 Introduction

Coupled partial differential systems with coupled boundary-value conditions are frequent in different areas of science and technology. Recently, an exact series solution for the homogeneous initial-value problem

\[ u_t(x,t) - Au_{xx}(x,t) = 0 \quad 0 < x < 1, \quad t > 0 \]  
\[ A_1 u(0,t) + B_1 u_x(0,t) = 0 \quad t > 0 \]  
\[ A_2 u(1,t) + B_2 u_x(1,t) = 0 \quad t > 0 \]  
\[ u(x,0) = f(x) \quad 0 \leq x \leq 1 \]  

where \( u = (u_1, u_2, \ldots, u_m)^T \) and \( f(x) = (f_1(x), f_2(x), \ldots, f_m(x))^T \) are a \( m \)-dimensional vectors, was constructed under the following hypotheses and notation:

1. The matrix coefficient \( A \) is a matrix which satisfies the following condition

\[ \text{Re}(z) > 0 \quad \forall z \in \sigma(A) \]  

where \( \sigma(C) \) denotes the set of all the eigenvalues of a matrix \( C \) in \( \mathbb{C}^{m \times m} \). Thus \( A \) is a positive stable matrix (where \( \text{Re}(z) \) denotes the real part of \( z \in \mathbb{C} \)).

2. Matrices \( A_i, B_i, i = 1, 2 \), are \( m \times m \) complex matrices, and we assume that the block matrix

\[ \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \text{ is regular} \]  

and also that the matrix pencil

\[ A_1 + \rho B_1 \text{ is regular.} \]  

Condition (7) is well known in the literature of singular systems of differential equations, see [1], and involves the existence of some \( \rho_0 \in \mathbb{C} \) so that matrix \( A_1 + \rho_0 B_1 \) is invertible. In this case, matrix \( A_1 + \rho B_1 \) is invertible with the possible exception of at most a finite number of complex numbers \( \rho \). In particular, we may assume that \( \rho_0 \in \mathbb{R} \).
Using condition (7) we can introduce the following matrices $\tilde{A}_1$ and $\tilde{B}_1$ defined by

$$
\tilde{A}_1 = (A_1 + \rho_0 B_1)^{-1} A_1, \quad \tilde{B}_1 = (A_1 + \rho_0 B_1)^{-1} B_1,
$$

which satisfy the condition $\tilde{A}_1 + \rho_0 \tilde{B}_1 = I$, where matrix $I$ denotes, as usual, the identity matrix. Under hypothesis (6), is it easy to show that matrix $B_2 - (A_2 + \rho_0 B_2) \tilde{B}_1$ is regular and we can introduce matrices $\tilde{A}_2$ and $\tilde{B}_2$ defined by

$$
\tilde{A}_2 = \left[B_2 - (A_2 + \rho_0 B_2) \tilde{B}_1\right]^{-1} A_2, \quad \tilde{B}_2 = \left[B_2 - (A_2 + \rho_0 B_2) \tilde{B}_1\right]^{-1} B_2,
$$

that satisfy the conditions $\tilde{B}_2 - (\tilde{A}_2 + \rho_0 \tilde{B}_2) \tilde{B}_1 = I, \tilde{B}_2 \tilde{A}_1 - \tilde{A}_2 \tilde{B}_1 = I$.

Under the above assumptions, the homogeneous problem (1)–(4) was solved in [2, 3] under two different cases:

(a) We can consider the following hypothesis:

exist $b_1 \in \sigma(\tilde{B}_1) - \{0\}, b_2 \in \sigma(\tilde{B}_2)$, and $v \in \mathbb{C}^m - \{0\}$, such that $$(\tilde{B}_1 - b_1 I) v = (\tilde{B}_2 - b_2 I) v = 0.$$

Then, if the vector valued function $f(x)$ satisfies hypotheses

$$
f \in \mathcal{C}^2([0,1]) \quad \begin{cases}
(1 - \rho_0 b_1) f(0) + b_1 f'(0) = 0 \\
- \frac{1 - b_2 + \rho_0 b_1 b_2}{b_1} f(1) + b_2 f'(1) = 0
\end{cases}
$$

with the additional condition:

$$f(x) \in \text{Ker} \left(\tilde{B}_1 - b_1 I\right) \cap \text{Ker} \left(\tilde{B}_2 - b_2 I\right), \quad 0 \leq x \leq 1$$

and

$$\text{Ker} \left(\tilde{B}_1 - b_1 I\right) \cap \text{Ker} \left(\tilde{B}_2 - b_2 I\right) \text{ is an invariant subspace with respect to matrix } A,$$

where a subspace $E$ of $\mathbb{C}^m$ is invariant by the matrix $A \in \mathbb{C}^{m \times m}$, if $A(E) \subset E$, we can construct an exact series solution $u(x, t)$ of homogeneous problem (1)–(4). This construction was made in Ref. [2].
(b) We can consider the following hypothesis:

\[ 0 \in \sigma \left( \tilde{B}_1 \right), \ a_2 \in \sigma \left( \tilde{A}_2 \right), \ \text{and we have } w \in \mathbb{C}^m - \{0\}, \]

so that \( \tilde{B}_1 w = \left( \tilde{A}_2 - a_2 I \right) w = 0 \). 

Then, if the vector valued function \( f(x) \) satisfies the hypotheses

\[
\begin{align*}
 f \in \mathcal{C}^2([0,1]) \\
f(0) &= 0 \\
a_2 f(1) + f'(1) &= 0
\end{align*}
\]

under the additional condition:

\[ f(x) \in \text{Ker} \left( \tilde{B}_1 \right) \cap \text{Ker} \left( \tilde{A}_2 - a_2 I \right), \ 0 \leq x \leq 1 \]

and

\[ \text{Ker} \left( \tilde{B}_1 \right) \cap \text{Ker} \left( \tilde{A}_2 - a_2 I \right) \text{ is an invariant subspace respect to matrix } A, \]

then we can construct an exact series solution \( u(x,t) \) of homogeneous problem (1)–(4). This construction was made in Ref. [3].

By other hand, the solution of the non-homogeneous problem

\[
\begin{align*}
 u_t(x,t) - A u_{xx}(x,t) &= G(x,t) , \ 0 < x < 1 , \ t > 0 \\
 A_1 u(0,t) + B_1 u_x(0,t) &= 0 , \ t > 0 \\
 A_2 u(1,t) + B_2 u_x(1,t) &= 0 , \ t > 0 \\
 u(x,0) &= f(x) , \ 0 \leq x \leq 1 
\end{align*}
\]

was made in Ref. [4] under the two different hypotheses (a) and (b).

This paper deals a method to construct the exact solution of the non-homogeneous problem with non-homogeneous conditions

\[
\begin{align*}
 u_t(x,t) - A u_{xx}(x,t) &= G(x,t) , \ 0 < x < 1 , \ t > 0 \\
 A_1 u(0,t) + B_1 u_x(0,t) &= P(t) , \ t > 0 \\
 A_2 u(1,t) + B_2 u_x(1,t) &= Q(t) , \ t > 0 \\
 u(x,0) &= f(x) , \ 0 \leq x \leq 1 
\end{align*}
\]
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in term of solutions of problems of the type (16)-(19). Throughout this paper we will assume the results and nomenclature given in [2, 3, 4]. This paper is organized as follows: In section 2 a method to construct a solution of (16)-(19) is obtained. In section 3 an algorithm and two illustrative examples are given. Conclusion are presented in section 4.

2 The proposed method

We consider the non-homogeneous problem with non-homogeneous conditions

\[ u_t(x,t) - Au_{xx}(x,t) = G(x,t), \quad 0 < x < 1, \quad t > 0 \] (20)

\[ A_1 u(0,t) + B_1 u_x(0,t) = P(t), \quad t > 0 \] (21)

\[ A_2 u(1,t) + B_2 u_x(1,t) = Q(t), \quad t > 0 \] (22)

\[ u(x,0) = f(x), \quad 0 \leq x \leq 1, \] (23)

where \( u(x,t), G(x,t), P(t), Q(t) \) and \( f(x) \) are vectors in \( \mathbb{C}^m \), and matrices \( A_1, A_2, B_1, B_2 \in \mathbb{C}^{m \times m} \) satisfying the conditions (5) and (6)-(7).

We are looking for a solution of (20)–(23) in the form

\[ u(x,t) = w(x,t) + v(x,t), \] (24)

where function \( v(x,t) \) satisfies the conditions

\[
\begin{align*}
A_1 v(0,t) + B_1 v_x(0,t) &= P(t), \quad t > 0, \\
A_2 v(1,t) + B_2 v_x(1,t) &= Q(t), \quad t > 0.
\end{align*}
\] (25)

Thus, we can define the function \( G_1(x,t) \) as

\[ G_1(x,t) = v_t(x,t) - Av_{xx}(x,t) \] (26)

then \( v(x,t) \) satisfies:

\[
\begin{align*}
v_t(x,t) - Av_{xx}(x,t) &= G_1(x,t), \quad 0 < x < 1, \quad t > 0 \\
A_1 v(0,t) + B_1 v_x(0,t) &= P(t), \quad t > 0 \\
A_2 v(1,t) + B_2 v_x(1,t) &= Q(t), \quad t > 0
\end{align*}
\] which implies that \( w(x,t) \) must satisfy
\[ w_t(x,t) - Aw_{xx}(x,t) = G(x,t) - G_1(x,t) \]
\[ = \tilde{G}(x,t), \quad 0 < x < 1, \quad t > 0 \]

with the homogeneous conditions:
\[
\begin{align*}
A_1w(0,t) + B_1w_x(0,t) &= 0, \quad t > 0, \\
A_2w(1,t) + B_2w_x(1,t) &= 0, \quad t > 0.
\end{align*}
\]
and the initial condition:
\[
\begin{align*}
w(x,0) &= f(x) - v(x,0) \\
&= \tilde{f}(x), \quad 0 \leq x \leq 1.
\end{align*}
\]

Then, function \( u(x,t) \) defined by (24) satisfy:
\[
\begin{align*}
&u_t(x,t) - Au_{xx}(x,t) = v_t(x,t) - Av_{xx}(x,t) + w_t(x,t) - Aw_{xx}(x,t) \\
&= G_1(x,t) + G(x,t) - G_1(x,t) \\
&= G(x,t),
\end{align*}
\]
with the boundary conditions (21)-(22) and the initial condition (23), so it is the desired solution of our problem (20)–(23).

Summarizing, the following theorem has been proved:

**Theorem 2.1** Let be consider the problem (20)–(23). Let \( v(x,t) \) be a vector valued function satisfying conditions (25). We define the vector valued functions
\[
\tilde{G}(x,t) = G(x,t) - G_1(x,t), \quad \tilde{f}(x) = f(x) - v(x,0),
\]
where \( G_1(x,t) \) is given by (26). We consider the non-homogeneous problem with homogeneous conditions

\[
\begin{align*}
w_t(x,t) - Aw_{xx}(x,t) &= \tilde{G}(x,t), \quad 0 < x < 1, \quad t > 0 \\
A_1w(0,t) + B_1w_x(0,t) &= 0, \quad t > 0 \tag{27} \\
A_2w(1,t) + B_2w_x(1,t) &= 0, \quad t > 0 \tag{28} \\
w(x,0) &= \tilde{f}(x), \quad 0 \leq x \leq 1 \tag{30}
\end{align*}
\]

which solution \( w(x,t) \) can be obtain using Theorem 2.1 of Ref. [4] if conditions established in this theorem holds. Then, \( u(x,t) = v(x,t) + w(x,t) \) is a solution of problem (20)-(23).
3 Algorithm and Examples

We can establish the following algorithm to solve problem (20)–(23):

**Algorithm 1** Solution of problem (20)-(23).

**Input data:** Matrices $A, A_1, A_2, B_1, B_2 \in \mathbb{C}^{m \times m}$, vectors $G(x), f(x) \in \mathbb{C}^m$.

**Result obtained:** If the stated assumptions are met, the series solution $u(x, t)$.

1. Determine a vector valued function $v(x, t)$ satisfying conditions (25).
2. Determine $\tilde{G}(x, t) = G(x, t) - G_1(x, t)$ and $\tilde{f}(x) = f(x) - v(x, 0)$.
3. Using the Algorithm given in Ref. [4] determine, if it is possible, a solution $w(x, t)$ of problem (27)–(30).
4. Determine the solution of problem (20)–(23) as $u(x, t) = w(x, t) + v(x, t)$.

Of course, the choice of the function $v(x, t)$ determine the choice of the functions $\tilde{G}(x, t)$ and $\tilde{f}(x)$, which must satisfy the hypotheses of Theorem 2.1 of Ref. [4], and depend on the nature of the given function $G(x, t)$. Here we present two different examples.

**Example 3.1** We consider problem (20)–(23) where function $G(x, t)$ is a linear combination of functions $\sin(\pi x)$ and $\cos(\pi x)$. Then, we will look for a function $v(x, t)$ which is also a linear combination of functions $\sin(\pi x)$ and $\cos(\pi x)$ with coefficients are functions of variable $t$. Thus, we look for a solution of (25) given in the form

$$v(x, t) = R_1(t) \sin(\pi x) + R_2(t) \cos(\pi x), \quad (31)$$

where vector-valued functions $R_i(t) \in C^1[0, +\infty), i = 1, 2$ must be determinate. This solution (31) must to satisfy boundary conditions (25):

$$A_1 v(0, t) + B_1 v_x(0, t) = P(t) \implies A_1 R_2(t) + \pi B_1 R_1(t) = P(t) \quad (32)$$

$$A_2 v(1, t) + B_2 v_x(1, t) = Q(t) \implies -A_2 R_2(t) - \pi B_2 R_1(t) = Q(t) \quad (33)$$

Writing (32) in matrix form:

$$\begin{pmatrix} A_1 & B_1 \\ -A_2 & -B_2 \end{pmatrix} \begin{pmatrix} R_2(t) \\ \pi R_1(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} \quad (33)$$

Premultiplying (33) by the invertible matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ one gets

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \begin{pmatrix} R_2(t) \\ \pi R_1(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -Q(t) \end{pmatrix} \quad (34)$$
Under hypothesis (6) this system has an unique solution. Thus, we have shown that we can determine a vector valued function $v(x,t)$ satisfying conditions (25) and defined by expression (31). Thus, we have now that

$$G_1(x,t) = v_t(x,t) - Av_{xx}(x,t)$$
$$= R'_1(t) \sin(\pi x) + R'_2(t) \cos(\pi x) + \pi^2 Av(x,t), \quad (35)$$

and we can apply Theorem 2.1. We will consider a concrete numerical example. Consider problem (20)-(23) where matrix $A \in \mathbb{C}^{4 \times 4}$ is given by

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & -2 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (36)$$

and the $4 \times 4$ matrices $A_i, B_i, i \in \{1, 2\}$, are

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (37)$$

The vectorial valued function $f(x)$ is defined by

$$f(x) = \begin{pmatrix} 0 \\ 0 \\ x^2 - 2x \\ 0 \end{pmatrix}, \quad (38)$$

function $G(x,t)$ is

$$G(x,t) = \begin{pmatrix} - \cos(\pi x) \left( \cos(t) + 2 \left( t + \pi^2 t^2 + \pi^2 \sin(t) \right) \right) + \frac{\sin(\pi x)}{\pi} \left( 2t \left( 1 + \pi^2 t \right) \right) \\ - \cos(\pi x) \left( t \left( 2 + 3\pi^2 t \right) + \pi^2 \sin(t) \right) + \pi t^2 \sin(\pi x) \\ e^{-t}(-1 + x)^2 x + \pi^2 \cos(\pi x) \left( t^2 + \sin(t) \right) - \pi t^2 \sin(\pi x) \\ 0 \end{pmatrix}, \quad (39)$$
and functions $P(t)$ and $Q(t)$ are defined by

$$P(t) = \begin{pmatrix} t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Q(t) = \begin{pmatrix} 0 \\ \sin(t) \\ 0 \\ 0 \end{pmatrix}. \quad (40)$$

We follow the Algorithm 1 step by step

1. We will determine a vector valued function $v(x,t)$ fulfilling conditions (25). As any of the components of the vector valued function $G(x,t)$ are combinations of functions $\sin(\pi x)$ and $\cos(\pi x)$, we will look for $v(x,t)$ in the form given by (31). To do this, from (34) we obtain

$$R_1(t) = \begin{pmatrix} t^2/\pi \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad R_2(t) = \begin{pmatrix} -t^2 - \sin(t) \\ -t^2 \\ -t^2 \\ 0 \end{pmatrix},$$

and thus determine the function $v(x,t)$ defined by

$$v(x,t) = \begin{pmatrix} -t^2 \cos(\pi x) - \cos(\pi x) \sin(t) + \frac{t^2 \sin(\pi x)}{\pi} \\ -t^2 \cos(\pi x) \\ 0 \\ 0 \end{pmatrix},$$

where replacing in (35) one gets

$$G_1(x,t) = \begin{pmatrix} -\cos(\pi x) \left( \cos(t) + 2 \left( t + \pi^2 t^2 + \pi^2 \sin(t) \right) \right) + \frac{2t (1 + \pi^2 t) \sin(\pi x)}{\pi} \\ -\cos(\pi x) \left( t (2 + 3\pi^2 t) + \pi^2 \sin(t) \right) + \pi t^2 \sin(\pi x) \\ \pi \left( \pi \cos(\pi x) \left( t^2 + \sin(t) \right) - t^2 \sin(\pi x) \right) \\ 0 \end{pmatrix}.$$  

Thus, vector valued function $v(x,t)$ verifies trivially (25).
2. From the definition of \( v(x,t) \) we determine \( \tilde{G}(x,t) \) and \( \tilde{f}(x) \):

\[
\tilde{G}(x,t) = \begin{pmatrix}
0 \\
0 \\
(x-1)^2xe^{-t} \\
0
\end{pmatrix},
\]

\[
\tilde{f}(x) = f(x) = \begin{pmatrix}
0 \\
0 \\
x^2-2x \\
0
\end{pmatrix}.
\]

3. Using the algorithm given in Ref. [4] we can construct a solution \( w(x,t) \) of problem (27)-(30) with these data. Observe that this problem is precisely the non-homogeneous problem with homogeneous conditions which was solved in the Example 3.2 of Ref. [4], whose exact solution is given by the series

\[
w(x,t) = \sum_{n \geq 0} \frac{-32e^{-\frac{(\pi+2n\pi)^2}{4}}t \sin \left(\frac{1}{2}(1+2k)\pi x\right)}{\pi^3(2k+1)^3} \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}
- 3072(-1)^n e^{-\frac{(2n+1)^2\pi^2}{4}} \left(\frac{e^{-\frac{1}{4}(2n+1)^2\pi^2}}{2} - 1\right) ((2n+1)^2\pi^2 - 10) \sin \left(\frac{(2n+1)\pi x}{2}\right) \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}.
\]

4. The solution of problem (20)-(23) is given by \( u(x,t) = w(x,t) + v(x,t) \), i.e., by the expression:

\[
u(x,t) = \sum_{n \geq 0} \frac{-32e^{-\frac{(\pi+2n\pi)^2}{4}}t \sin \left(\frac{1}{2}(1+2k)\pi x\right)}{\pi^3(2k+1)^3} \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix}
- 3072(-1)^n e^{-\frac{(2n+1)^2\pi^2}{4}} \left(\frac{e^{-\frac{1}{4}(2n+1)^2\pi^2}}{2} - 1\right) ((2n+1)^2\pi^2 - 10) \sin \left(\frac{(2n+1)\pi x}{2}\right) \begin{pmatrix}
0 \\
0 \\
1 \\
0
\end{pmatrix} \\
+ \begin{pmatrix}
-t^2 \cos (\pi x) - \cos (\pi x) \sin (t) + \frac{t^2 \sin (\pi x)}{\pi} \\
-t^2 \cos (\pi x) \\
0 \\
0
\end{pmatrix}.
\]
Example 3.2 We consider problem (20)–(23). Suppose that \( G(x,t) \) is a polynomial in \( x \), with coefficients are functions of the variable \( t \). Thus, we look for a vector valued function \( v(x,t) \) which is also a polynomial in \( x \) (cubic, for example), whose coefficients are functions of the variable \( t \), in the form

\[
v(x,t) = R_3(t)x^3 + R_2(t)x^2 + R_1(t)x + R_0(t),
\]

where functions \( R_i(t) \in C^1[0, +\infty), i = 0, 1, 2, 3 \) must be determinate. This function \( (41) \) satisfy the boundary conditions (25), i.e.

\[
\begin{align*}
A_1R_0(t) + B_1R_1(t) &= P(t) \\
A_2(R_3(t) + R_2(t) + R_1(t) + R_0(t)) + B_2(3R_3(t) + 2R_2(t) + R_1(t)) &= Q(t).
\end{align*}
\]

we can write the above system in matrix form:

\[
\begin{pmatrix}
A_1 & B_1 & 0 & 0 \\
A_2 & A_2 + B_2 & A_2 + 2B_2 & A_2 + 3B_2
\end{pmatrix}
\begin{pmatrix}
R_0(t) \\
R_1(t) \\
R_2(t) \\
R_3(t)
\end{pmatrix}
= \begin{pmatrix}
P(t) \\
Q(t)
\end{pmatrix}.
\]

(42)

Taking block matrices

\[
\hat{A} = \begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & -3I & I & 0 \\
0 & 2I & 0 & I
\end{pmatrix},
\hat{B} = \begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 3I & I & 0 \\
0 & -2I & 0 & I
\end{pmatrix},
\]

which trivially satisfy that \( \hat{A}\hat{B} = I \), (42) can be written in the form

\[
\begin{pmatrix}
A_1 & B_1 & 0 & 0 \\
A_2 & A_2 + B_2 & A_2 + 2B_2 & A_2 + 3B_2
\end{pmatrix}
\hat{A}\hat{B}
\begin{pmatrix}
R_0(t) \\
R_1(t) \\
R_2(t) \\
R_3(t)
\end{pmatrix}
= \begin{pmatrix}
P(t) \\
Q(t)
\end{pmatrix},
\]

thus

\[
\begin{pmatrix}
A_1 & B_1 & 0 & 0 \\
A_2 & B_2 & A_2 + 2B_2 & A_2 + 3B_2
\end{pmatrix}
\begin{pmatrix}
R_0(t) \\
R_1(t) \\
3R_1(t) + R_2(t) \\
R_3(t) - 2R_1(t)
\end{pmatrix}
= \begin{pmatrix}
P(t) \\
Q(t)
\end{pmatrix}.
\]

(43)

We can rewrite (43) in the form
\[
\begin{pmatrix}
A_1 & B_1 \\
A_2 & B_2
\end{pmatrix}
\begin{pmatrix}
R_0(t) \\
R_1(t)
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
A_2 + 2B_2 & A_2 + 3B_2
\end{pmatrix}
\begin{pmatrix}
R_2(t) + 3R_1(t) \\
R_3(t) - 2R_1(t)
\end{pmatrix} = \begin{pmatrix}
P(t) \\
0
\end{pmatrix}.
\]

(44)

If we impose the condition:

\[
\begin{pmatrix}
0 & 0 \\
A_2 + 2B_2 & A_2 + 3B_2
\end{pmatrix}
\begin{pmatrix}
R_2(t) + 3R_1(t) \\
R_3(t) - 2R_1(t)
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

or equivalently:

\[
\begin{aligned}
R_2(t) &= -3R_1(t) \\
R_3(t) &= 2R_1(t)
\end{aligned}
\]

(45)

from (44) we have the matrix block system

\[
\begin{pmatrix}
A_1 & B_1 \\
A_2 & B_2
\end{pmatrix}
\begin{pmatrix}
R_0(t) \\
R_1(t)
\end{pmatrix} = \begin{pmatrix}
P(t) \\
0
\end{pmatrix}.
\]

(46)

Taking into account (6), system (46) have an unique solution. Thus, we have shown that we can determine a vector valued function \(v(x,t)\) satisfying conditions (25) and defined by expression (31). Thus, we have now that

\[
G_1(x,t) = v_t(x,t) - Av_{xx}(x,t)
\]

\[
= R_3'(t)x^3 + R_2'(t)x^2 + R_1'(t)x + R_0'(t) - A(6R_3(t)x + 2R_2(t))
\]

(47)

and we can apply Theorem 2.1. We will consider a concrete numerical example. Consider problem (20)-(23) where matrix \(A \in \mathbb{C}^{4 \times 4}\) is given by

\[
A = \begin{pmatrix}
2 & 0 & 0 & -1 \\
1 & 2 & 1 & -2 \\
-1 & 0 & 2 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(48)

and the matrices \(A_i, B_i, i \in \{1, 2\}\) given by (37). Also, the vectorial valued functions \(f(x)\) and \(G(x,t)\) will be defined respectively as

\[
f(x) = \begin{pmatrix}
0 \\
x^2 - 1 \\
0 \\
0
\end{pmatrix},
\]

(49)

and
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\[ G(x,t) = \begin{pmatrix} \cos(t) - 2t + 12t^2 + 2tx - 24t^2x - 6tx^2 + 4tx^3 \\ -2t + 6t^2 - 12t^2x + e^{-t}x^3 - 2e^{-t}x^4 + e^{-t}x^5 \\ -6t^2 + 12t^2x \\ 0 \end{pmatrix}, \] (50)

and functions \( P(t) \) and \( Q(t) \) defined by (40).

**WE FOLLOW THE ALGORITHM 1 STEP BY STEP**

1. **We will determine a vector valued function \( v(x,t) \) fulfilling conditions (25).** As any of the components of the vector valued function \( G(x,t) \) are polynomials in the variable \( x \), with coefficients are functions of the variable \( t \), we will look for \( v(x,t) \) in the form given by (41). To do this, from (46) we obtain

\[
R_0(t) = \begin{pmatrix} \sin(t) - t^2 \\ -t^2 \\ 0 \\ 0 \end{pmatrix}, \quad R_1(t) = \begin{pmatrix} t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

and from (45) we obtain

\[
R_2(t) = \begin{pmatrix} -3t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad R_3(t) = \begin{pmatrix} 2t^2 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

and therefore we have the function

\[
v(x,t) = \begin{pmatrix} -t^2 + t^2x - 3t^2x^2 + 2t^2x^3 + \sin(t) \\ -t^2 \\ 0 \\ 0 \end{pmatrix}.
\]

From (47) one gets

\[
G_1(x,t) = \begin{pmatrix} -2t + 12t^2 + 2tx - 24t^2x - 6tx^2 + 4tx^3 + \cos(t) \\ -2t + 6t^2 - 12t^2x \\ -6t^2 + 12t^2x \\ 0 \end{pmatrix}.
\]

Thus, vector valued function \( v(x,t) \) verifies trivially (25).
2. From the definition of \( v(x, t) \) we determine \( \tilde{G}(x, t) \) and \( \tilde{f}(x) \):

\[
\tilde{G}(x, t) = \begin{pmatrix}
0 \\
(x - 1)^2 x^3 e^{-t} \\
0 \\
0
\end{pmatrix},
\]

\[
\tilde{f}(x) = f(x) = \begin{pmatrix}
0 \\
x^2 - 1 \\
0 \\
0
\end{pmatrix}.
\]

3. Using the algorithm given in Ref. [4] we can construct a solution \( w(x, t) \) of problem (27)-(30) with these data. Observe that this problem is precisely the non-homogeneous problem with homogeneous conditions which was solved in the Example 3.1 of Ref. [4], whose exact solution is given by the series

\[
w(x, t) = \sum_{n \geq 0} -32(-1)^n e^{-\frac{1}{2}((2n+1)^2 \pi^2 - 1)} \left\{ -2 + (2n+1)^2 \pi^2 \right\} \frac{A(n)}{\pi^3(2n+1)^3} \left( \begin{array}{c}
0 \\
1 \\
0 \\
0
\end{array} \right)
\]

\[
+ \sum_{n \geq 0} -64 e^{-\frac{(2n+1)^2 \pi^2}{2}} \left( e^{-\frac{(2n+1)^2 \pi^2}{2}} - 1 \right) A(n) \cos \left( \frac{(2n+1) \pi x}{2} \right) \left( \begin{array}{c}
0 \\
1 \\
0 \\
0
\end{array} \right)
\]

where

\[
A(n) = (480 + (2n+1)\pi(2n+1)(-1)^n + (2n+1)\pi((-1)^n(2n+1)\pi - 6))).
\]

4. The solution of problem (20)-(23) is given by \( u(x, t) = w(x, t) + v(x, t) \), i.e., by the expression:

\[
u(x, t) =
\]
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\[ \left( \sum_{n \geq 0} -32(-1)^n e^{-\frac{1}{2}(\pi+2n\pi)^2} \cos \left( \frac{1}{2}(2n+1)\pi x \right) \right) \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) \]

\[ + \left( \sum_{n \geq 0} -32(-1)^n e^{-\frac{1}{2}(\pi+2n\pi)^2} \cos \left( \frac{1}{2}(2n+1)\pi x \right) \right) \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) \]

\[ + \left( \sum_{n \geq 0} -32(-1)^n e^{-\frac{1}{2}(\pi+2n\pi)^2} \cos \left( \frac{1}{2}(2n+1)\pi x \right) \right) \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) \]

where

\[ \mathcal{A}(n) = (480+(2n+1)\pi(-144(-1)^n+(2n+1)\pi((-1)^n(2n+1)\pi-6))). \]

4 Conclusion

In this paper a method to solve non-homogeneous problem with non-homogeneous conditions of the type (20)-(23) in terms of the solution of a non-homogeneous with homogeneous conditions problem (16)-(19) with appropriate parameters, is developed. The computational process is outlined in Algorithm 1. The choose of the appropriate function \( v(x,t) \) is illustrated in the examples 3.1 and 3.2.

References


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