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Influence of Selected Parameters on Oscillatory Behaviour of Dynamical System

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Abstract

The paper deals with differential equation of the second order containing a small parameter at the highest derivative, studying various oscillation patterns occurring in the system, numerically modelling the results in MATLAB built-in solver using the Runge-Kutta method. The mathematical base is provided in the theory of singular perturbations, briefly described as well.

Mathematics Subject Classification: 34E15, 34C60, 93B35

Keywords: singular perturbation theory, ode solvers, numerical modelling and simulation

1 Singular Perturbation Theory

Singular perturbation theory deals with the study of problems having a prominent parameter for which the problem solutions at a limiting parameter value are significantly different from the solutions of the general problem. Singular perturbation problem solutions relating on differential equations frequently depend on a number of broadly different length or time scales [2]. The most popular as well as the mostly understood methods for solving such problems are the method of matched asymptotic expansions [3], [4] and the method of multiple-scale analysis [5]. The difference in using these methods in cases that both of them can be possibly applied, with show that the latter one may often be preferable for solving singularly perturbed problems is summarized in [1].

The basic difference between these two methods, as [6] describes, is that while under the method of matched asymptotic expansions the solution is found in different regions being patched together to form a composite expansion, under the multiple-scale analysis the solution is found as a generalized version of a composite expansion. The coordinates for each domain are introduced with the new variables being considered as independent. Thus the original ordinary differential equation is transformed into a partial differential equation or into a system of algebraic-differential equations and, generally speaking, a single variable can be replaced by an infinite sequence of independent time scales.

2 Multiple-Scale Analysis

In fact, regarding the multiple-scale analysis problems, there are generally two time scales being introduced, although, additional one or two scales can be considered vital in some cases.

As of the real-world applications, slow-fast systems can be found in numerous natural processes, such as climate systems (e.g. El Nino phenomenon) [7], in celestial mechanics (e.g. the Einstein equation for Mercury) [8], enzyme kinetics [2] and in many others. Recently, multiple-scale analysis was used for vibration analysis of piezoelectrically actuated microbeam with non-ideal boundary conditions [9] or for nonlinear vibration analysis of mechanical systems with multiple joint clearances [10], in which a sliding pendulum is subjected to analysis. In [11], the method of multiple scales serves for primary resonance analysis of dynamic stability and nonlinear vibrations in a rotor system both with flexible and rigid blades. Stability and bifurcation analysis for the Kaldor-Kalecki model of business cycle with a discrete delay and a distributed delay is studied in [12], where the method is used for finding the explicit formulae determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions.

3 Singularly Perturbed Dynamical System Example

As of the examination of the influence of selected parameters on oscillatory behaviour of dynamical system, this paper follows the most commonly used approach, so there are two time scales being introduced. Thus, in the following text, t stands for slow time and τ for fast time, being interrelated as $\tau = \frac{t}{\varepsilon}$, where $\varepsilon \to 0^+$ is the singular perturbation parameter.

The nonlinear dynamical system described by the parameter-dependent second-order differential equation being subjected to the analysis is of the form:

$$\begin{gathered} \varepsilon^2 y'' + f(t, y) = 0, \\ y(-\delta) = y_0, y'(-\delta) = y_1, \end{gathered} \tag{1}$$

where y_0 and y_1 are the initial conditions, $y_\varepsilon = (., y_0, y_1)$ is a direct output, t is time, ε is the singular perturbation parameter, $0 < \varepsilon \ll 1$ and $f(t, y)$ is defined as:

$$f(t,y) = \begin{cases} y^{4n+1} & \text{for } t \in \langle -\delta, 0 \rangle, \quad (2) \\ y \displaystyle\prod_{i=1}^{2n} (y^2 - {h_i}^2(t)) & \text{for } \ t \in (0, \infty), \quad (3) \end{cases}$$

where $\delta > 0$, $n \in \mathbb{N}$ and $h_i(t)$ is a positive continuous function on $(0, \infty)$, and for $i = 1, 2, \ldots, 2n$ $h_i(t)$ takes the form:

$$\mathrm{h}_i(t) = (it + (\cos t - 1)^i). \tag{4}$$

The approach leading to the conclusion that the theory of singular perturbations can be used for solving the problem consists of rewriting the equation (1) into the system of three autonomous equations of the first order, applying the limit $\varepsilon \to 0^+$ for the newly established algebraic-differential reduced system, followed by rewriting the same equation (1) using the new variable $\tau = \dfrac{t}{\varepsilon}$, applying the same limit for the newly established system and getting the so-called associated system, resulting in the conclusion that the algebraic-differential reduced system and the associated system agree when $\varepsilon \neq 0$, but differ significantly in the limit when $\varepsilon = 0$ regarding the level of phase space structure. The details of applying this approach are described, e.g. in [13].

4 Numerical Simulation and Results

Let us presume that for (1) - (4), $n = 1$. Therefore, the equations

$$f(t,y)=0 \tag{5}$$

and

$$w=0 \tag{6}$$

have, for f,w one solution, if $t^* \in \langle -\delta, 0)$ and five solutions for $t^* \in \langle 0, \infty)$, which gives us a critical manifold with multiple arms. As the eingenvalues of the Jacobian are

$$\lambda(t^*,y,w) = \pm\sqrt{-\frac{\partial f}{\partial y}(t^*,y)}, \tag{7}$$

the associated system has, in case of $n=1$, one equilibrium for $t^* \in (-\delta, 0)$ and five equilibria for $t^* \in (0,\infty)$. It follows, that the two parts of the critical manifold S, corresponding the saddles of the associated system are normally hyperbolic submanifolds [13].

From the point of view of energy, the total energy functional is obtained in the form:

$$\begin{aligned} H(t,y,w) &= \tfrac{1}{2}w^2 + V(t,y), \\ V(t,y) &= \int_0^y f(t,s)ds, \end{aligned} \tag{8}$$

with potential $V(t,y)$ and multiple barriers and wells as a result of multiplying (1) by velocity y', integrating the multiplied problem followed by using the initial conditions. The function H is the Hamiltonian and is consisted of the sum of the kinetic and potential energies for the system. To characterize the trajectories of the algebraic-differential reduced system, the level surfaces $H(t,y,w) = H(t)$, defined in (t,y,w)-space for $w \in \mathbb{R}$ as

$$w = \pm\sqrt{2(H(t) - V(t,y))} \tag{9}$$

are used.

Computing the derivative of H along the solution of the algebraic-differential reduced system for $n=1$ and $t>0$ leads to the following solution:

$$\begin{aligned} H'(t) &= w_\varepsilon(t)w'_\varepsilon(t) + f(t,y_\varepsilon(t))y'_\varepsilon(t) \\ &\quad + \int_0^{y_\varepsilon(t)} \frac{\partial f}{\partial t}(t,s)ds \\ &= h_1h_2{y_\varepsilon}^2\left[(h_1h_2)' - \frac{1}{2}\left(\frac{h'_1}{h_2} + \frac{h'_2}{h_1}\right){y_\varepsilon}^2\right]. \end{aligned} \tag{10}$$

The figures representing the numerical modelling of the system (1) using the built-in MATLAB solver ode45 based on an implicit Runge-Kutta formula summarise the conclusions resulting from calculations.

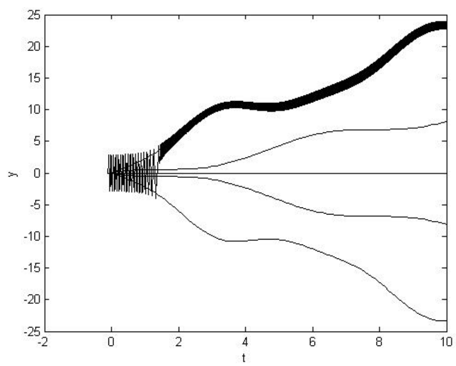

Figure 1: Numerical solution of (1), where $y_0(-0.1) = 0$, $y_1(-0.1) = 1$, $\varepsilon = 0.07$, $n = 1$, $h_i(t) = (it + (\cos t - 1)^i)$, $i = 1, 2$.

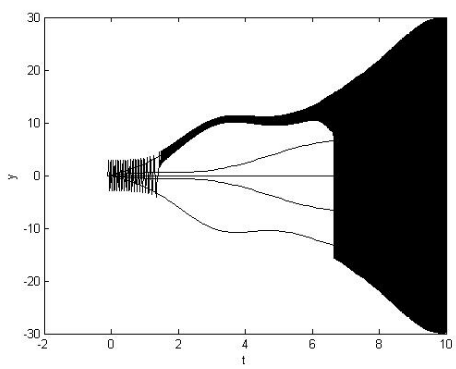

Figure 2: Numerical solution of (1), where $y_0(-0.1) = 0$, $y_1(-0.1) = 1$, $\varepsilon = 0.071$, $n = 1$, $h_i(t) = (it + (\cos t - 1)^i)$, $i = 1, 2$.

5 Conclusion

The presented figures show some of the oscillatory patterns occurring in the analysed nonlinear dynamical system as a result of change in the values of the singular perturbation parameter and the initial conditions. The conclusions are drawn from comparing numerous additional results, so the figures serve only as an illustrative example to conclude that with the increasing values of initial conditions, the initial phase of the oscillations is extended, as well as

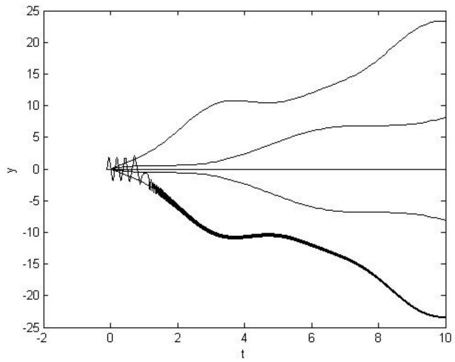

Figure 3: Numerical solution of (1), where $y_0(-0.1) = 0.1$, $y_1(-0.1) = 0.3$, $\varepsilon = 0.09$, $n = 1$, $h_i(t) = (it + (\cos t - 1)^i)$, $i = 1, 2$.

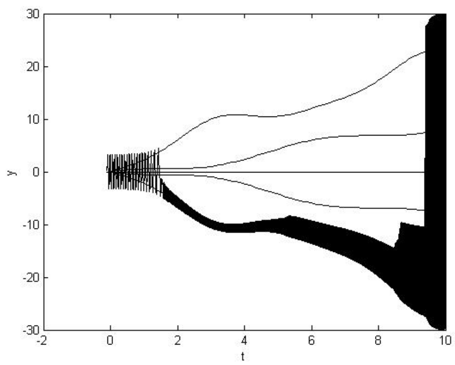

Figure 4: Numerical solution of (1), where $y_0(-0.1) = 0.2$, $y_1(-0.1) = 1.8$, $\varepsilon = 0.09$, $n = 1$, $h_i(t) = (it + (\cos t - 1)^i)$, $i = 1, 2$.

that with $\varepsilon \to 0^+$ the frequency of oscillations increases. The system does not have to be stabilized on one of the arms of the critical manifold, as results from Figures 2 and 4, as the energy of the system along the trajectory of the direction of motion is not constant.

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