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On a Nonlocal (Two Point) Boundary Value Problem of a Nonlinear Functional Integro-Differential Equation of Arbitrary (Fractional) Order

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Abstract

The functional equations, functional integral equations and functional differential equations have many applications in nonlinear analysis. Also, the fractional order integral equations and fractional order differential equations have many applications in mathematical physics. Here we study the existence of solutions of a functional integral equations of arbitrary (fractional) order in the two classes of continuous and integrable functions. As an application we study the existence of solutions of a nonlocal (two point) boundary value problem of a nonlinear functional integro-differential equation of arbitrary (fractional) order.

Keywords: Fractional-order integro-differential equation, nonlocal boundary value problem, fixed point theorem

1 Introduction

Let $\alpha,\ \beta \in (0,1]$, and D^{α}, I^{β} are the fractional order derivative (in the Caputo sense) and fractional order integral (see [15]-[17]).

Let $C[0,1]$, $AC[0,1]$ and $L^1[0,1]$ be the classes of continuous, absolutely continuous and integrable functions on the interval $[0,1]$ respectively ([12]-[14]).

Let $\alpha \in (0,1]$ and consider the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s) g(s, I^{1-\alpha} y(s) ds)), \qquad t \in [0,1] \tag{1}$$

and the nonlocal (two point) boundary value problems of the arbitrary (fractional) order functional integro-differential equations

$$x'(t) = f(t, \int_0^1 k(t,s) g(s, D^{\alpha} x(s) ds)), \qquad t \in (0,1) \tag{2}$$

and

$$x'(t) = f(t, \int_0^1 k(t,s) g(s, D^{\alpha} x(s) ds)), \qquad a.e., \quad t \in (0,1) \tag{3}$$

with the nonlocal, two point, boundary condition

$$x(\tau) = \gamma\, x(\xi), \qquad \tau \in [0,1), \qquad \xi \in (0,1]\ , \gamma \neq 1. \tag{4}$$

The existence of solutions of the nonlinear functional integral equations and the nonlocal boundary and initial value problems have been studied by some authors, see for example [1]-[4], [11], [13] and [18] and [5]-[10] respectively .

Our main object here is to study, under the suitable assumptions, the existence of solutions $y \in C[0,1]$ and $y \in L^1[0,1]$ of the functional integral equation (1). As an application we study the existence of solutions of the nonlocal (two point) boundary value problems (2)-(4) or (3)-(4).

2 Functional integral equation

Consider the functional integral equation (1)

$$y(t) = f(t, \int_0^1 k(t,s) g(s, I^{1-\alpha} y(s) ds)), \qquad t \in [0,1].$$

Here we prove the existence of solutions $y \in C[0,T]$ and $y \in L^1[0,T]$ of the functional integral equation (1).

2.1 Existence of continuous solution

Consider the following of assumptions

(i) $f : I = [0,1] \times R \longrightarrow R$ is continuous and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq L\,|x-y|, \ \forall\, t \in I, \ x,y \in R$$

(ii) $g : I \times R \longrightarrow R$ is continuous and satisfies the Lipschitz condition

$$|g(t,x) - g(t,y)| \leq h\,|x-y|, \ \forall\, t \in I, \ x,y \in R.$$

(iii) $k : I \times I \to R$ is continuous in $t \in I$ for every $s \in I$ and measurable in $s \in I$ for all $t \in I$ such that $\sup_{t\in I} \int_0^1 |k(t,s)|ds \leq M$.

Theorem 2.1 Let the assumptions (i) - (iii) be satisfied. If $\frac{L\,h\,M}{\Gamma(2-\alpha)} \leq 1$, then the functional integral equation (1) has a unique solution $y \in C[0,1]$.

Proof. Define the operator F associated with the functional integral equation (1) by

$$Fy(t) = f(t, \int_0^1 k(t,s)g(s, I^{1-\alpha}\, y(s)ds)), \qquad t \in I. \tag{5}$$

The operator F maps $C\,[0,1]$ in to it self, for this let $y \in C[0,1]$, $t_1, t_2 \in I$, $t_1 < t_2$ and $|\ t_2 - t_1\ | \leq \delta$, then

$$\begin{aligned}
|Fy(t_2) - Fy(t_1)| &= |f(t_2, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds) - f(t_1, \int_0^1 k(t_1,s)g(s, I^{1-\alpha}y(s))ds)| \\
&= |f(t_2, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds) - f(t_1, \int_0^1 k(t_1,s)g(s, I^{1-\alpha}y(s))ds) \\
&+ f(t_1, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds) - f(t_1, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds)| \\
&\leq |f(t_2, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds) - f(t_1, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds)| \\
&+ |f(t_1, \int_0^1 k(t_2,s)g(s, I^{1-\alpha}y(s))ds) - f(t_1, \int_0^1 k(t_1,s)g(s, I^{1-\alpha}y(s))ds)|.
\end{aligned}$$

This proves that $F : C\,[0,1] \to C\,[0,1]$.

Now to prove that F is contraction we have following, let $x_1, x_2 \in C\,[0,1]$, then

$$\begin{aligned}
|Fx_2(t) - Fx_1(t)| &= |f(t, \int_0^1 k(t,s)g(s, I^{1-\alpha}x_2(t)ds)) \\
&- f(t, \int_0^1 k(t,s)g(s, I^{1-\alpha}x_1(t)ds))| \\
&= |f(t, \int_0^1 k(t,s)g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, x_2(\theta)d\theta ds)) \\
&- f(t, \int_0^1 k(t,s)g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, x_1(\theta)d\theta ds))|
\end{aligned}$$

$$
\begin{aligned}
&\leq L \,|\int_0^1 k(t,s)g(s,\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ x_2(\theta)d\theta ds) \\
&- \int_0^1 k(t,s)g(s,\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ x_1(\theta)d\theta ds)| \\
&\leq L\ h \int_0^1 |k(t,s)||\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ x_2(\theta)d\theta ds \\
&- \int_0^1 |k(t,s)||\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ x_1(\theta)d\theta ds| \\
&\leq L\ h \int_0^1 |k(t,s)|ds\ \frac{s^{1-\alpha}}{\Gamma(2-\alpha)}|\ x_2(\theta) -\ x_1(\theta)| \\
&\leq \frac{Lh\ M}{\Gamma(2-\alpha)}\ \|x_2 - x_1\|,
\end{aligned}
$$

then

$$\|Fx_2 - Fx_1\| \ \leq\ \frac{L\ h\ M}{\Gamma(2-\alpha)}\ \|x_2 - x_1\|.$$

If $\frac{L\ h\ M}{\Gamma(2-\alpha)} \leq\ 1$, then F is a contraction and by using Banach fixed point theorem [12]-[14] there exists a unique solution $y \in\ C\ [0,1]$ of the functional integral equation (1).

2.2 Existence of integrable solution

Consider the following of assumptions

(i^*) $f : I\ \times\ R \to\ R$ is measurable in t and satisfies the Lipschitz condition

$$|f(t,x)\ -\ f(t,y)|\ \leq\ L\ |x-y| \quad \forall\ t \in\ I,\quad x,y\ \in\ R$$

and $\int_0^1\ f(t,0)\ dt\ \leq\ r_1$

(ii^*) $g : I\ \times\ R \to\ R$ is measurable in t and satisfies the Lipschitz condition

$$|g(t,x)\ -\ g(t,y)|\ \leq\ h\ |x-y| \quad \forall\ t \in\ I,\quad x,y\ \in\ R$$

and $\int_0^1\ g(t,0)\ dt\ \leq\ r_2$.

Theorem 2.2 Let the assumptions (i^*), (ii^*) and (iii) be satisfied. If $\frac{L\ h\ M}{\Gamma(2-\alpha)} \leq 1$, then the functional integral equation (1) has a unique solution $y \in\ L^1[0,1]$.
Proof. Define the operator G associated with the integral equation (3) by

$$Gy(t) = f(t,\int_0^1 k(t,s)g(s,I^{1-\alpha}\ y(s)ds)), \qquad t \in\ I. \tag{6}$$

The operator G maps $L^1\ [0,1]$ in to it self, for this let $y \in\ L^1\ (I)$, then

$$|Gy(t)| = |f(t, \int_0^1\ k(t,s)g(s, I^{1-\alpha}\ y(s)ds))|\ .$$

But from assumption (i^*) we are deduce that

$$|f(t,y)| - |f(t,0)| \leq\ |f(t,y) - f(t,0)| \leq\ L|y|$$

which implies that

$$|f(t,y)| \leq L|y| + |f(t,0)|.$$

Also

$$|g(t,y)| \leq h|y| + |g(t,0)|\ .$$

Now

$$|f(t,y)| \leq\ L|\int_0^1\ k(t,s)g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ y(\theta)d\theta ds) + |f(t,0)|.$$

Integrating we obtain

$$\begin{aligned}
\int_0^1 |Gy(t)|dt \ &\leq\ \int_0^1\ L\ |\int_0^1\ k(t,s)g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ y(\theta)d\theta ds)|dt + r_1 \\
&\leq\ L\ \int_0^1\ \int_0^1\ |k(t,s)||g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ y(\theta)d\theta)|dsdt + r_1 \\
&\leq\ L\ \int_0^1\ |k(t,s)|dt \int_0^1\ g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ y(\theta)d\theta)ds + r_1 \\
&\leq\ L\ M\ |\int_0^1 g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ y(\theta)d\theta ds)| + r_1 \\
&\leq\ L\ M\ \int_0^1 h \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ |y(\theta)|d\theta ds + r_2 d\theta + r_1 \\
&\leq\ L\ M\ h\ \int_0^1 \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ |y(\theta)|d\theta ds + r_2 + r_1 \\
&\leq\ L\ M\ h\ \int_0^1 (\int_\theta^1 \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} ds|y(\theta)|d\theta + r_2 + r_1 \\
&\leq\ L\ M\ h\ \int_0^1 |y(\theta)|\frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)} d\theta + r_2 + r_1 \\
&\leq\ L\ M\ h\ \|y\|\frac{1}{\Gamma(2-\alpha)} + r_2 + r_1.
\end{aligned}$$

This prove that $G:\ L^1\ [0,1] \to\ L^1\ [0,1]$.
Now to prove that G contraction we have the following.

Let $x_1, x_2 \in L^1[0,1]$, then

$$\begin{aligned}
|Gx_2(t) - Gx_1(t)| &= |f(t, \int_0^1 k(t,s) g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta) d\theta) ds)) \\
&- f(t, \int_0^1 k(t,s) g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta) d\theta) ds)| \\
&\leq L \int_0^1 |k(t,s)||g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta) d\theta) ds) \\
&- g(s, \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta) d\theta) ds)|, \\
&\leq L\, h \int_0^1 |k(t,s)|| \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta) d\theta ds \\
&- \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta) d\theta ds|,
\end{aligned}$$

then

$$\begin{aligned}
\int_0^1 |G\, x_2(t) - G\, x_1(t)| dt &\leq \int_0^1 Lh| \int_0^1 k|(t,s)|| \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta) d\theta ds - \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta) d\theta ds| dt \\
&\leq \int_0^1 Lh| \int_0^1 k|(t,s)| dt| \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta) d\theta ds - \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta) d\theta ds| \\
&\leq LhM \int_0^1 | \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, x_2(\theta) d\theta) ds| - \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, x_1(\theta) d\theta) ds| \\
&\leq LhM| \int_0^1 \int_\theta^1 \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta) ds) d\theta - \int_0^1 \int_\theta^1 \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, x_1(\theta) ds) d\theta\ | \\
&\leq L\ h\ M| \int_0^1 \frac{(1-\theta)^{(1-\alpha)}}{\Gamma(2-\alpha)}\, x_2(\theta) d\theta - \int_0^1 \frac{(1-\theta)^{(1-\alpha)}}{\Gamma(2-\alpha)}\, x_1(\theta) d\theta\ | \\
&\leq LhM\ \frac{(1-\theta)^{(1-\alpha)}}{\Gamma(2-\alpha)} \int_0^1 |\ x_2(\theta) - x_1(\theta)| d\theta \\
&\leq \frac{L\ h\ M}{\Gamma(2-\alpha)} \|x_2 - x_1\|.
\end{aligned}$$

If $\frac{L\ h\ M}{\Gamma(2-\alpha)} \leq 1$, then G is contraction and by using Banach fixed point theorem [12]-[14] there exists a unique solution $y \in L^1[0,1]$ of the functional integral equation (1).

3 Boundary value problem

Now we study the existence of solution of the problems (2)-(4) and (3)-(4).
Theorem 3.1 Let the assumptions of Theorem (2.1) be satisfied, then the nonlocal boundary value problem (2)-(4) has a unique solution $x \in C[0,1]$.

Proof. Let $\frac{dx}{dt} = y \in C[0,1]$ in (2), then we obtain

$$x(t) = x(0) + \int_0^t y(s)ds \in C[0,1] \tag{7}$$

where y is the solution of the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)g(s, I^{1-\alpha} y(s)ds)) \in C[0,1].$$

Using the nonlocal boundary condition (4) we obtain

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^\xi y(s)ds,$$

then

$$x(0) + \int_0^\tau y(s)ds = \gamma\, x(0) + \gamma \int_0^\xi y(s)ds$$

and

$$x(0) = \frac{\gamma}{1-\gamma} \int_0^\xi y(s)ds - \frac{1}{1-\gamma} \int_0^\tau y(s)ds.$$

Hence

$$x(t) = \frac{\gamma}{1-\gamma} \int_0^\xi y(s)ds - \frac{1}{1-\gamma} \int_0^\tau y(s)ds + \int_0^t y(s)ds. \tag{8}$$

From Theorem (2.1), there exists a unique solution $y \in C[0,1]$ satisfying the functional integral equation (1), then there exists a unique solution $x \in C[0,1]$ of the problems (2)-(4) giving by (7).

Now for the problem (3)-(4) we can prove the following theorem.
Theorem 3.2 Let the assumptions of Theorem (2.2) be satisfied, then from Theorem 2.1 the nonlocal boundary value problem (3)-(4) has a unique solution $x \in AC[0,1]$.
Proof. Let $\frac{dx}{dt} = y \in L^1[0,1]$ in (3), then we obtain

$$x(t) = x(0) + \int_0^t y(s)ds \in AC[0,1] \tag{9}$$

where y is the solution of the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)g(s, I^{1-\alpha} y(s)ds)) \in L^1[0,1].$$

Using the nonlocal boundary condition (4) we obtain

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^\xi y(s)ds,$$

then

$$x(0) + \int_0^\tau y(s)ds = \gamma\ x(0) + \gamma \int_0^\xi y(s)ds$$

and

$$x(0) = \frac{\gamma}{1-\gamma} \int_0^\xi y(s)ds - \frac{1}{1-\gamma} \int_0^\tau y(s)ds.$$

Hence

$$x(t) = \frac{\gamma}{1-\gamma} \int_0^\xi y(s)ds - \frac{1}{1-\gamma} \int_0^\tau y(s)ds + \int_0^t y(s)ds \in AC[0,1]. \quad (10)$$

From Theorem (2.2), there exists a unique solution $y \in L^1 [0,1]$ satisfying the functional integral equation (1), then there exists a unique solution $x \in AC [0,1]$ of the problems ((3)-(4) given by (9).

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