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A Note on Some Fixed Point Theorems for Set-valued Non-self Mappings in Banach Spaces

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Abstract

Let C be a nonempty weakly compact convex separable subset of a Banach space X satisfying property (D). The aim of this paper is to show that every nonexpansive and $1-\chi-$contractive mapping $T : C \to KC(X)$ satisfying an inwardness condition has a fixed point, where $KC(X)$ is the family of all nonempty compact convex subsets of X. Our result is an extension of some results given by Benavides and Ramírez [4] and some other authors.

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1 Introduction

In 1969, Nadler extended the Banach Contraction Principle to multivalued contraction mappings in complete metric spaces. In 1974, Lim [10] used Edelstein's method of asymptotic center to prove that every multivalued nonexpansive self mapping $T : E \to K(E)$ has a fixed point where E is a nonempty bounded closed convex subset of a uniformly convex Banach space X. In 1990, Kirk and Massa [8] extend Lim's theorem to assure the existence of a fixed point of a multivalued nonexpansive self mapping $T : E \to KC(E)$ where E is a nonempty bounded closed convex subset of a Banach space X which has the property that the asymptotic center in E of each bounded sequence of X is nonempty and compact. In 2001, H.K. Xu [11] extended Kirk and Massa's theorem to nonself mapping $T : E \to KC(X)$ which satisfying an inwardness condition.

Recently, Dompongsa et. al [2] prove that if X is a reflexive Banach space satisfying the property (D) and E be a weakly compact convex subset of X, then every nonexpansive mapping $T : E \to KC(E)$ has a fixed point.

The purpose of this paper we show that if X is a Banach space satisfying property (D) and C be a nonempty weakly compact convex separable subset of X, then every nonexpansive and $1-\chi-$contractive mapping $T : C \to KC(X)$ satisfying an inwardness condition has a fixed point. Our result is an extension of some results given by Benavides and Ramı́rez [4] and Theorem 3.5 in [2].

2 Preliminaries

Let X be a Banach space and C a nonempty subset of X. We shall denote by $CB(X)$ the family of nonempty bounded closed subsets of X, by $K(X)$ the family of nonempty compact subsets of X, and by $KC(X)$ the family of nonempty compact convex subsets of X. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $CB(X)$, i.e.,

$$H(A,B) = \max\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\}, \quad A,B \in CB(X),$$

where $\operatorname{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B. A set-valued mapping $T : C \to CB(X)$ is said to be a contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \le k\|x-y\|, \quad x,y \in E. \tag{1}$$

If (1) is valid when $k = 1$, then T is called nonexpansive. A point x is a fixed point for a set-valued mapping T if $x \in Tx$.

Recall that the Hausdorff measure of noncompactness of a nonempty bounded subset B of X is defined as the number:

$$\chi(B) = \inf\{d > 0 : B \text{ can be covered by finitely many balls of radii} \le d\}.$$

A set-valued mapping $T : E \to 2^X$ is called $\chi-$condensing (resp. $1-\chi-$contractive) if, for each bounded subset B of E with $\chi(B) > 0$, there holds the inequality

$$\chi(T(B)) < \chi(B) \text{ (resp. } \chi(T(B)) \le \chi(B)).$$

Here $T(B) = \cup_{x\in B}Tx$.

In order to study the fixed point theory for non-self mappings, we must introduce some terminology for boundary condition. The inward set of C at $x \in C$ defined by

$I_C(x) := \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}$.

Clearly $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C does. A set-valued mapping $T : C \to 2^X\{\emptyset\}$ is said to be inward on C if

$Tx \subset I_C(x) \ \forall x \in C$.

Let $\bar{I}_C(x) := x + \{\lambda(z - x) : z \in C, \lambda \geq 1\}$. Note that for a convex C, we have $\bar{I}_C(x) = \overline{I_C(x)}$, and T is said to be weakly inward on C if

$Tx \subset \bar{I}_C(x) \ \forall x \in C$.

Let C be a nonempty bounded closed subset of Banach spaces X and $\{x_n\}$ bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf \left\{ \limsup_n \|x_n - x\| : x \in C \right\}, \\ A(C, \{x_n\}) &= \left\{ x \in C : \limsup_n \|x_n - x\| = r(C, \{x_n\}) \right\}. \end{aligned}$$

If D is a bounded subset of X, the *Chebyshev radius* of D relative to C is defined by

$$r_C(D) := \inf \left\{ \sup\{\|x - y\| : y \in D\} : x \in C \right\}.$$

Obviously, the convexity of C implies that $A(C, \{x_n\})$ is convex. Notice that $A(C, \{x_n\})$ is a nonempty weakly compact set if C is weakly compact, or C is a closed convex subset of a reflexive Banach spaces X.

Definition 2.1 *Let $\{x_n\}$ and C be a nonempty bounded closed subset of Banach spaces X. Then $\{x_n\}$ is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.*

The property (D) introduced in [2].

Definition 2.2 (Dhompongsa et. al [2]) *A Banach space X is said to satisfy property (D) if there exists $\lambda \in [0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \subset E$ which is regular asymptotically uniform relative to E, and any sequence $\{y_n\} \subset A(E, \{x_n\})$ which is regular asymptotically uniform relative to E we have*

$$r(E, \{y_n\}) \leq \lambda r(E, \{x_n\}). \tag{2}$$

The method of asymptotic centers play an important role in the fixed point theory of both single- and set-valued nonexpansive mappings, due to the fundamental lemma below.

Lemma 2.3 (Geobel[5] and Lim[9]). *Let $\{x_n\}$ and C be as above. Then we have*

(i) There always exists a subsequence of $\{x_n\}$ which is regular with respect to C;

(ii) if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to C.

The following result are now basic in the fixed point theorem for set-valued mappings.

Lemma 2.4 [1, Deimling] *Let E be a nonempty bounded closed convex subset of a Banach space X and $T : E \to FC(X)$ an upper semicontinuous and $\chi-$condensing mapping. Assume $Tx \cap \overline{I_E(x)} \neq \varnothing$ for all $x \in E$. Then T has a fixed point.*

3 The Result

We begin this section with an extension of Benavides and Ramírez [3]. Our result also improves the work of Dhompongsa et al. Theorem 3.6 in [2], in the sense that we will extends this theorem to non-self mappings.

In order to prove our first result, we need the following Lemma which is prove along the proof of Kirk-Massa theorem as it appear in [11].

Lemma 3.1 (Kumam and Plubtieng [7]) *Let C be a nonempty closed bounded convex separable subset of a Banach space X. $T : C \to KC(X)$ is a nonexpansive such that $T(C)$ is a bounded set and which satisfies $Tx \subset I_C(x)$, $\forall x \in C$, $\{x_n\}$ is a sequence in C such that $\lim_n d(x_n, Tx_n) = 0$. Then there exist a subsequence $\{z_n\}$ of $\{x_n\}$ such that $Tx \cap I_A(x) \neq \emptyset, \forall x \in A := A(C, \{z_n\})$.*

With this observation, we are able to prove our main result. The proof below is inspired by same ideas as in the proof of Theorem 3.6 in [2] and Theorem 3.4 [4].

Theorem 3.2 *Let C be a nonempty weakly compact convex separable subset of a reflexive Banach spaces X which satisfies property (D), and $T : C \to$*

$KC(X)$ be a nonexpansive and 1-χ-contractive mapping, such that satisfies the inwardness condition:

$$Tx \subset I_C(x) \quad \textit{for all } x \in C.$$

Then T has a fixed point.

Proof Let $x_0 \in C$ be fixed and set, for each $n \geq 1$, define

$$T_n(x) = \frac{1}{n}x_0 + \left(\frac{n-1}{n}\right)Tx,$$

for each $x \in C$. Then T_n is set-valued contraction. Bearing in mind that for each $x \in I_C(x)$ is convex and contains C, it is easily see that $T_n x \subset I_C(x)$ for all $x \in C$. Since $T_n(\cdot)$ is 1-χ-contractive mapping, it follows by [3, pp.382] that $T_n(\cdot)$ is χ-condensing. Hence, by Lemma 2.4, $T_n(\cdot)$ has a fixed point $x_n \in C$. Thus we have a sequence $\{x_n\}$ in C such that $dist(x_n, Tx_n) \leq \frac{1}{n} diamC \to 0$, as $n \to \infty$. By Lemma 2.3, we can assume that sequence $\{x_n\} \subset C$ is a regular asymptotically uniform relative to C. Denote $A = A(C, \{x_n\})$, and $r = r(C, \{x_n\})$. Since X satisfies the property (D) with corresponding $\lambda \in [0, 1)$, we have

$$r(A, \{y_n\}) \leq \lambda r(A, \{x_n\}). \tag{1}$$

Fix $x_1 \in A$ and for each $n \in \mathbb{N}$, the contraction $T_n^1 : A \to KC(X)$ defined by

$$T_n^1(x) = \frac{1}{n}x_1 + \left(\frac{n-1}{n}\right)Tx, \quad x \in A.$$

By Lemma 3.1, we have $Tx \cap I_A(x) \neq \emptyset, \ \forall x \in A$. Since $I_A(x)$ is convex, it follow that $T_n^1 x \cap I_A(x) \neq \emptyset, \ \forall x \in A$. Again, since $T_n(\omega, \cdot)$ is 1-χ-contractive mapping, it follows by [3, pp.382] that $T_n(\omega, \cdot)$ is χ-condensing. Hence, by Lemma 2.4, $T_n(\omega, \cdot)$ has a fixed point $x_n^1 \in A$. Consequently, we can get a sequence $\{x_n^1\}$ in A satisfying $d(x_n^1, Tx_n^1) \to 0$ as $n \to \infty$. Again, applying Lemma 3.1, we obtain

$$Tx \cap I_{A^1}(x) \neq \emptyset \ \forall x \in A^1 = A(C, \{x_n^1\}).$$

By Lemma 2.3, we can assume that sequence $\{x_n^1\} \subset A$ is a regular asymptotically uniform relative to C. Again, by property (D) with a corresponding $\lambda \in [0, 1)$, we have

$$r(A, \{x_n^1\}) \leq \lambda r(A, \{x_n\}). \tag{2}$$

By induction, for each $m \geq 1$, we take a sequence $\{x_n^m\}_n \subseteq A^{m-1}$ such that $\lim_n d(x_n^m, Tx_n^m) = 0$. We construct the set $A^m := A(C, \{x_{n_\alpha}^m\})$ which is regular asymptotically uniform relative to A, and

$$r(A, \{x_n^m\}) \leq \lambda^m r(A, \{x_n\}). \tag{3}$$

Now we can proceed the proof as in Theorem 3.6 in [2] and Theorem 3.4 [4] to obtain a fixed point. This completes the proof. □

Corollary 3.3 (Theorem 3.6 [2]) *Let C be a nonempty weakly compact convex separable subset of a reflexive Banach spaces X which satisfies property (D). If $T : C \to KC(C)$ is a nonexpansive mapping. Then T has a fixed point.*

Proof It follows immediately from Theorem 3.2, since every self set-valued mappings satisfies the inwardness condition. □

Theorem 3.4 *Let C be a nonempty weakly compact convex separable subset of a reflexive Banach spaces X which satisfies property (D), and $T : C \to KC(X)$ be a nonexpansive such that satisfying the nonstrict Opial condition. If T satisfies*

$$Tx \subset I_C(x) \quad \text{for all } x \in C,$$

then T has a fixed point.

Proof This follows immediately from [3, Theorem 4.5] and Theorem 3.2, since nonstrict Opial condition implies 1-χ-contractive map (see [3]). □

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